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Naturalness of the Space of States in Quantum Mechanics

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We show how certain constructions of quantum mechanics, like monopoles, instantons, and the Schrödinger–von Neumann equation, are related to geometric functors which are representable. We study the differential geometry of the projective bundle associated with an infinite-dimensional separable Hilbert space, and we construct a universal connection which is described as a subspace of skew-Hermitian operators. This connection is responsible for the Berry phase.

1. INTRODUCTION

As is well known, quantum mechanics (Dirac, 1958; Bohm, 1993b) is an essentially complex theory based on the principle of superposition of amplitudes, which are elements of a Hilbert space that is usually separable and infinite dimensional, though many quantum systems of physical interest, such as, for example, spin systems, involve finite-dimensional space.

The discovery of geometric phases (Berry, 1994) reinforced the idea that quantum mechanics is a theory with deep geometric and topological roots; in particular, it was shown (Aharonov and Anandan, 1987; see also Simon, 1983) that the geometric phase in the finite-dimensional case is given by the holonomy of the Narasimhan–Ramanan connection. The extension to infinite dimensions was considered only at the formal level through the inductive limit of finite-dimensional spaces.

In the present paper, after discussing the origin and naturalness of the geometrical structures involved in the description of the quantum states, we present a rigorous treatment of the geometry of the infinite-dimensional projective Hilbert bundle and its universal connection.

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We use the term "naturalness" in a technical sense, that is, in the categorical framework. This is done in Section 2. The main idea is that starting from the empty set, the successive application of certain representable functors (Mac Lane and Birkoff, 1979) leads in a completely canonical way to the mathematical constructions used in quantum mechanics: complexification and quaternization of real vector spaces, Clifford algebras, principal G-bundles, connections and spaces of states (projective spaces).

In Section 3 we review how the universal connection of Narasimhan and Ramanan (1961) is responsible for the time evolution of the Aharonov–Anandan (1987) wave function and therefore of geometric phases, through parallel transport along the solution of the Schrödinger–von Neumann equation in the finite-dimensional projective space. For simplicity we restrict ourselves to the nondegenerate case.

In Section 4, which is the longest of this paper, we study the differential geometry of the projective Hilbert bundle associated with an infinite-dimensional separable Hilbert space $\mathcal{H}$. We define a differentiable structure on this bundle using homogeneous spaces of Banach Lie groups. This allows us to define a universal connection in terms of a certain subspace of skew-Hermitian operators on $\mathcal{H}$. We also show that this subspace carries a complex structure. Finally, in the Appendix we prove a key technical result needed to study homogeneous spaces.

2. NATURALNESS

2.1. Representation of Functors

We assume here that the reader is familiar with the concepts of categories and functors (Mac Lane and Birkoff, 1979).

Let φ: $\mathcal{C} \to Set$ be a functor (cofunctor) with $\mathcal{C}$ an arbitrary category and Set the category of sets. A universal object (Mac Lane and Birkoff, 1979) for φ is a pair (C_0, a_0) where C_0 is an object in $\mathcal{C}$ and a_0 is an element of $\varphi(C_0)$ which solves the following problem: for any object C in $\mathcal{C}$ and any element a in $\varphi(C)$ there exists and is unique an arrow α: $C_0 \to C$ (α: $C \to C_0$) with functorial image $\alpha_*(\alpha^*)$ such that $a = \alpha_*(a_0)$ [$a = \alpha^*(a_0)$]. This definition is depicted in the following diagrams:

$$\varphi: \quad \mathcal{C} \to Set$$

$$\begin{array}{ccccc} C & \longrightarrow & \varphi(C) & \ni & a \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha_*} & & \uparrow \\ C_0 & \longrightarrow & \varphi(C_0) & \ni & a_0 \end{array} \qquad\qquad \begin{array}{ccccc} C & \longrightarrow & \varphi(C) & \ni & a \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\alpha^*} & & \uparrow \\ C_0 & \longrightarrow & \varphi(C_0) & \ni & a_0 \end{array}$$

Universal object for a functor — Universal object for a cofunctor

It can be easily proved that if a universal object exists, then, up to isomorphism, it is unique. In fact, let (C_0, a_0) and (C_0', a_0') be universal objects for φ (covariant). Then a_0' induces a unique α_0: $C_0 \to C_0'$ such that $a_0' = \alpha_{0*}(a_0)$ and a_0 a unique β_0: $C_0' \to C_0$ such that $a_0 = \beta_{0*}(a_0')$; then $a_0' = (\alpha_0 \circ \beta_0)_*(a_0')$ and $a_0 = (\beta_0 \circ \alpha_0)_*(a_0)$, and by uniqueness $\alpha_0 \circ \beta_0 = id_{C_0'}$ and $\beta_0 \circ \alpha_0 = id_{C_0}$, so $\alpha_0 = \beta_0^{-1}$ is an isomorphism in $\mathscr{C}$. Notice that the isomorphism is itself canonical. For contravariant functors the proof is similar.

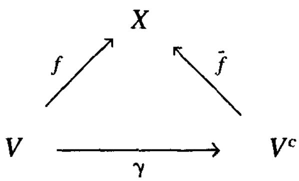
If $H(C_0, C)$ denotes the morphisms from C_0 to C, then for each C in $\mathscr{C}$ the assignment

$$\psi_{\mathscr{C}_0,a_0}(C)\colon \quad \varphi(C) \to H(C_0, C)$$

given by $\psi_{C_0,a_0}(C)(a) = \alpha$ is a bijection of sets [for a confunctor, $H(C_0, C)$ is replaced by $H(C, C_0)$], i.e., it is an isomorphism in *Set*. It is clear that C_0 is a distinguished object in $\mathscr{C}$.

The following examples are important for later applications.

(i) *Complexification.* Given a real vector space V, one defines the functor φ_V: $\mathscr{V}ect_{\mathbb{C}} \to Set$ through $\varphi_V(X) = \mathrm{Lin}_{\mathbb{R}}(V, X)$ (real-linear transformations from V to X) and α_*: $\mathrm{Lin}_{\mathbb{R}}(V, X) \to \mathrm{Lin}_{\mathbb{R}}(V, Y)$, $\alpha_*(r) = \alpha \circ r$ if $\alpha \in \mathrm{Lin}_{\mathbb{C}}(X, Y)$. A universal object is a complex vector space V^c together with a real-linear transformation γ: $V \to V^c$ such that for any complex vector space X and any real-linear map f: $V \to X$ there exists and is unique a complex-linear map $\bar{f}$: $V^c \to X$ such that the following diagram commutes:



The correspondences with the general definition of a universal object are $C_0 = V^c$, $a_0 = \gamma$, $a = f$, and $\alpha = \bar{f}$. The pair (V^c, γ) is the complexification of the real vector space V and is given by $V^c = V \oplus V$ $(= V^{\oplus 2})$ with

$$(v \oplus v') + (w \oplus w') = (v + w) \oplus (v' + w')$$

$$(\alpha + i\beta)(v \oplus v') = (\alpha v - \beta v') \oplus (\alpha v' + \beta v)$$

for $\alpha, \beta \in \mathbb{R}$ and $i \in \mathbb{C}$, and $\gamma(v) = v \oplus 0$; for real-linear f, $\bar{f}(v \oplus v') = f(v) + if(v')$.

(ii) *Quaternization.* A (left) quaternization of a real vector space V is a left $\mathbb{H}$-module V^q together with a real-linear transformation κ: $V \to V^q$ such that for any left $\mathbb{H}$-module Y and any real-linear map f: $V \to Y$ there exists and is unique a transformation of left $\mathbb{H}$-modules $\bar{f}$: $V^q \to Y$ such that $\bar{f} \circ \kappa = f$. It is easy to verify that the pair (V^q, κ) is a universal object for the functor $\psi_V: {}_{\mathbb{H}}\mathcal{M} \to \mathit{Set}$ given by $\psi_V(Y) = \mathrm{Lin}_{\mathbb{R}}(V, Y)$ and $\psi_V(\alpha)$: $\mathrm{Lin}_{\mathbb{R}}(V, X) \to \mathrm{Lin}_{\mathbb{R}}(V, Y)$, $\psi_V(\alpha)(s) = \alpha \circ s$ if $\alpha \in_{\mathbb{H}} H(X, Y)$; the functorial diagram is the following:

$$\begin{array}{ccccc} Y & \longrightarrow & \mathrm{Lin}_{\mathbb{R}}(V, Y) & \ni & f \\ \bar{f}\uparrow & & \uparrow \bar{f}_* & & \uparrow \\ V^q & \longrightarrow & \mathrm{Lin}_{\mathbb{R}}(V, V^q) & \ni & \kappa \end{array}$$

The explicit formulas for the quaternization of V are $V^q = V^{\oplus 4}$ with sum and multiplication by reals as in the complex case, left multiplication of $v_1 \oplus v_2 \oplus v_3 \oplus v_4$ by i, j, and k in $\mathbb{H}$, respectively, given by $(-v_2) \oplus v_1 \oplus (-v_4) \oplus v_3$, $(-v_3) \oplus v_4 \oplus v_1 \oplus (-v_2)$, and $(-v_4) \oplus (-v_3) \oplus v_2 \oplus v_1$, and $\kappa(v) = v \oplus 0 \oplus 0 \oplus 0$; for real-linear f,

$$\bar{f}(v_1 \oplus v_2 \oplus v_3 \oplus v_4) = f(v_1) + if(v_2) + jf(v_3) + kf(v_4).$$

(iia) *Quaternization of a complex vector space.* A complex vector space is quaternized along the same lines as a real vector space is complexified. Namely, let V be a complex vector space. A (left) quaternization of V is a left $\mathbb{H}$-module V^q together with a complex-linear transformation ρ: $V \to V^q$ such that for any left $\mathbb{H}$-module W and any complex-linear map f: $V \to W$ there exists and is unique a transformation of left $\mathbb{H}$-modules $\bar{f}$: $V^q \to W$ such that $\bar{f} \circ \rho = f$. The pair (V^q, ρ) defined by $V^q = V^{\oplus 2}$ with sum as in the complex case and left multiplication by $\mathbb{H}$ given by $(z_1 + jz_2)(v_1 \oplus v_2) = (z_1v_1 - z_2v_2) \oplus (z_1v_2 + z_2v_1)$, and $\rho(v) = v \oplus 0$, such that for any complex-linear f, $\bar{f}(v \oplus v') = f(v) + jf(v')$ is a universal object for the functor $\varphi_V: {}_{\mathbb{H}}\mathcal{M} \to \mathit{Set}$ with $\varphi_V(Y) = \mathrm{Lin}_{\mathbb{C}}(V, Y)$ and $\varphi_V(\alpha) \equiv \alpha_*$: $\mathrm{Lin}_{\mathbb{C}}(V, X) \to \mathrm{Lin}_{\mathbb{C}}(V, Y)$, $\varphi_V(\alpha)(s) = \alpha \circ s$ if $\alpha \in_{\mathbb{H}} H(X, Y)$. The functorial diagram is

$$\begin{array}{ccccc} W & \longrightarrow & \mathrm{Lin}_{\mathbb{C}}(V, W) & \ni & f \\ \bar{f}\uparrow & & \uparrow \bar{f}_* & & \uparrow \\ V^q & \longrightarrow & \mathrm{Lin}_{\mathbb{C}}(V, V^q) & \ni & \rho \end{array}$$

with $\bar{f}_*(\rho) = \bar{f} \circ \rho$.

(iii) *Clifford Algebra.* A Clifford algebra (Lawson and Michelsohn, 1989) for a real or complex vector space V with an inner product g is a pair consisting of an associative algebra $Cl_{V,g}$ with unit 1_C and a linear transformation γ: $V \to Cl_{V,g}$ satisfying $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2g(v, w)1_C$, i.e., a Clifford map, such that for any associative algebra A with unit 1_A and any Clifford map f:

$V \to A$ there exists a unique algebra homomorphism $\bar{f}$: $Cl_{V,g} \to A$ such that $\bar{f} \circ \gamma = f$. $(Cl_{V,g}, \gamma)$ is a universal object for the functor

$$\varphi_{V,g}: \{\text{associative algebras with unit over the field } k \ (k = \mathbb{R} \text{ or } \mathbb{C})\} \to Set$$

given by $\varphi_{V,g}(A) = \{\text{Clifford maps } V \to A\} = \{\alpha: V \to A \text{ linear}, \alpha(v)\alpha(v') + \alpha(v')\alpha(v) = g(v, v')1_A\}$ on objects, and $\varphi_{V,g}(\lambda: A_1 \to A_2) \equiv \lambda_*$: {Clifford maps $V \to A_1$} $\to$ {Clifford maps $V \to A_2$}, $\lambda_*(r) = \lambda \circ r$ on morphisms. This is illustrated in the following diagram, with $\bar{f}_*(\gamma) = \bar{f} \circ \gamma$:

$$\begin{array}{ccccc} A & \longrightarrow & Cliff(V, A) & \ni & f \\ \bar{f}\big\uparrow & & \big\uparrow \bar{f}_* & & \big\uparrow \\ Cl_{V,g} & \longrightarrow & Cliff(V, Cl_{V,g}) & \ni & \gamma \end{array}$$

The concept of universal object is closely related to that of *representable functor*, which we now explain. Let φ and ψ be covariant functors from the category $\mathscr{C}$ to the category $\mathscr{D}$. A *natural transformation* between φ and ψ, Φ: $\varphi \to \psi$, is a rule that to each object X in $\mathscr{C}$ assigns a morphism $\Phi(X)$: $\varphi(X) \to \psi(X)$ in $\mathscr{D}$ such that for any morphism f: $X \to Y$ in $\mathscr{C}$ the following diagram conmutes:

$$\begin{array}{ccc} \varphi(X) & \xrightarrow{\varphi(f)} & \varphi(Y) \\ \Phi(X)\big\downarrow & & \big\downarrow\Phi(Y) \\ \psi(X) & \underset{\psi(f)}{\longrightarrow} & \psi(Y) \end{array}$$

i.e., $\Phi(Y) \circ \varphi(f) = \psi(f) \circ \Psi(X)$. (For contravariant functors φ and ψ the horizontal arrows are inverted.) If for any object Z in $\mathscr{C}$, $\Phi(Z)$ is an isomorphism, then the natural transformation is called *natural equivalence*. (A well-known example of natural transformation is when φ and ψ are the contravariant functors of p and $p + 1$ differential forms, Ω^p and Ω^{p+1}, respectively, from the category of differentiable manifolds to the category of real vector spaces; then $\Phi = d$ is the De Rham exterior derivative.) It can be shown that if (C_0, a_0) is a universal object for the covariant functor φ: $\mathscr{C} \to Set$, then

$$\Psi_{C_0,a_0}: \quad \varphi \to H(C_0, —)$$

is a natural equivalence; Ψ_{C_0,a_0} is called a *representation* of φ, and φ is called representable. [For a cofunctor ψ: $\mathscr{C} \to Set$, $H(C_0, —)$ is replaced by $H(—, C_0)$.] Notice the close analogy with the idea of representation of groups. Here $H(C_0, —)$ is the *canonical functor* $\mathscr{C} \to Set$ given by $H(C_0, X)$ on objects and λ_*: $H(X_0, X_1) \to H(X_0, X_2)$, $\lambda_*(f) = \lambda \circ f$ on morphisms.

For the examples (i), (ii), and (iii) above, the representations of the functors φ_V, ψ_V, and $\varphi_{V,g}$ are given by $\Psi_{V^c,\gamma}(X)(f) = \bar{f}$, $\Psi_{V_q,\kappa}(Y)(f) = \bar{f}$, and $\Psi_{Cl_{V,g}}, \gamma(A)(f) = \bar{f}$, respectively. The inverse functions are $\Psi_{V^c,\gamma}(X)^{-1}(h) = h \circ \gamma$, $\Psi_{V^q,\kappa}(Y)^{-1}(h) = h \circ \kappa$, and $\Psi_{Cl_{V,g}}, \gamma(A)^{-1}(h) = h \circ \gamma$, with h, respectively, in $H(V^c, X)$, $H(V^q, Y)$, and $H(Cl_{V,g}, A)$.

2.2. Principal Bundles, Connections, and Parallel Transport

Let G be a Lie group. A smooth principal G-bundle (Aguilar, 1996; Kobayashi and Nomizu, 1963) consists of two manifolds P and X together with a smooth function $\pi: P \to X$ (respectively the total space, the base space, and the projection), and a smooth action $\psi: P \times G \to P$, $\psi(p, g) = \psi_g(p)$, such that there exists a set of diffeomorphisms called local trivializations φ: $U \times G \to \pi^{-1}(U)$ obeying $\pi(\varphi(x, g)) = x$ and $\varphi(x, g) = \psi(\varphi(x, e), g)$ (e is the identity element and the set of all U's is an open cover of X). These data are collectively denoted by $\xi = (P, \pi, X, G, \psi)$ or more simply by $\xi = P(X, G)$, and diagramatically represented by $\xi: G \to P \xrightarrow{\pi} X$. One can show that the action of G on P is free, transitive on $P_x \equiv \pi^{-1}(\{x\})$ for each $x \in X$ and $\pi(p_1) = \pi(p_2)$ if and only if there exists $g \in G$ such that $p_2 = p_1 g$. We call G the fiber of the bundle because at each x, P_x is diffeomorphic to G.

Principal G-bundles can also be defined over an arbitrary topological space X, just by asking the functions in the previous definition to be only continuous and G only a topological group. A principal G-bundle over X is *numerable* if there is a partition of unity subordinated to the open cover of X determined by the local trivializations. In particular, principal G-bundles over paracompact spaces (e.g., differentiable manifolds) are numerable.

Let $\xi = P(X, G)$ be a smooth G-bundle, $p \in P$ and $x = \pi(p)$. Then the smooth one-to-one function $\alpha_p: G \to P$, $g \mapsto \alpha_p(g) := \psi(p, g)$ induces a diffeomorphism between G and $P_x = \pi^{-1}(\{x\})$, the fiber over x given by $\tilde{\alpha}_p: G \to P_x$, $\tilde{\alpha}_p(g) := \alpha_p(g)$. As a consequence for each $g \in G$ one has the exact sequence of real vector spaces (tangent spaces)

$$0 \to T_gG \xrightarrow{(d\alpha_p)_g} T_{pg}P \xrightarrow{(d\pi)_{pg}} T_xX \to 0 \tag{2.1}$$

i.e., $(d\alpha_p)_g$ and $(d\pi)_{pg}$ are linear maps, $(d\alpha_p)_g$ is one-to-one, $(d\pi)_{pg}$ is onto, and $\ker(d\pi)_{pg} = im(d\alpha_p)_g$ [where $(d\alpha_p)_g$ and $(d\pi)_{pg}$ are the differentials of the functions α_p and π at the points g and pg, respectively]. Then one has the isomorphisms of vector spaces ρ_1: $T_{pg}P/im((d\alpha_p)_g) \to T_xX$ given by $\rho_1([v_{pg}]) = (d\pi)_{pg}(v'_{pg})$ for any $v'_{pg} \in [v_{pg}] = v_{pg} + (d\alpha_p)_g(T_gG)$, and ρ_2: $T_gG \oplus T_xX \to T_{pg}P$ given by $\rho_2(v_g \oplus w_x) = (d\alpha_p)_g(v_g) + \gamma(w_x)$, where γ: $T_xX \to T_{pg}P$ satisfies $(d\pi)_{pg} \circ \gamma = id_{T_xX}$ and is the linear extension of the (noncanonical) map which associates a vector in $T_{pg}P$ to each basis vector of T_xX. (The existence of the second isomorphism is known as the excision property of an exact sequence of vector spaces.) Since e is a canonical element of G, the pair (ξ, p) induces the exact sequence

$$0 \to \mathcal{G} \xrightarrow{(d\alpha_p)_e} T_pP \xrightarrow{(d\pi)_p} T_xX \to 0 \tag{2.2}$$

where $\mathcal{G} \equiv T_eG$ is the Lie algebra of G.

A G-bundle morphism $\xi' = P'(X', G) \to \xi = P(X, G)$ is a topological group homomorphism h: $G \to G$ together with continuous functions f: $P' \to P$ and $\bar{f}$: $X' \to X$ such that $\psi \circ (f \times h) = f \circ \psi'$ and $\pi \circ f = \bar{f} \circ \pi'$ (the bar in $\bar{f}$ indicates that $\bar{f}$ is determined by f). If $X' = X$, $\bar{f} = id_X$, and $h = id_G$, then f is a homeomorphism and $\xi' \to \xi$ is a G-bundle isomorphism.

Let $\mathcal{B}_G(X)$ be the set of isomorphism classes of numerable principal G-bundles over X (an element of this set is denoted by $[\xi]$, the equivalence class of ξ) and $\mathcal{B}^s_G(X)$ the set of isomorphism classes of smooth principal G-bundles over a manifold X. One can show that there is a bijection $\mathcal{B}^s_G \to \mathcal{B}_G$ (when G is a Lie group with at most countable many components). A theorem due to Milnor (1956) shows that for any topological group G (e.g., a Lie group) there exists a *universal bundle*

$$\xi_G: \quad G \to PG \xrightarrow{\pi_G} BG$$

such that for any space X there is a bijection between $\mathcal{B}_G(X)$ and $[X, BG]$, the set of homotopy classes of maps from X to BG. (G does not determine uniquely the universal bundle, but only its homotopy type.) So for any numerable G-bundle ξ over X there is a continuous function α: $X \to BG$ such that the pullback bundle $\alpha^*\xi_G$: $G \to \alpha^*PG \xrightarrow{\pi} X$ is in the class of ξ. The total space of $\alpha^*\xi_G$ is the subspace of $X \times PG$ given by the pairs (x, p) such that $\alpha(x) = \pi_G(p)$. If G is a compact Lie group, then ξ_G is filtered by smooth principal G-bundles

$$\xi^r_G: G \to P_rG \xrightarrow{\pi_r} B_rG$$

such that $\cup_{r=1}^{\infty} B_rG = BG$.

It can be verified that for any topological group G the pair $(BG, [\xi_G])$ is a universal object for the contravariant functor φ_G: $\mathcal{HT}op \to Set$ defined by $\varphi_G(X) = \mathcal{B}_G(X)$ on objects, and $\varphi_G([f]: X \to Y) = [f^*]: \mathcal{B}_G(Y) \to \mathcal{B}_G(X)$, $[f]^*([\xi]) = [f^*\xi]$ on morphisms. The following diagram illustrates this fact:

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{B}_G(X) & \ni & [\xi] \\ {\scriptstyle [f]}\downarrow & & \uparrow{\scriptstyle [f]^*} & & \uparrow \\ BG & \longrightarrow & \mathcal{B}_G(BG) & \ni & [\xi_G] \end{array}$$

The representation of φ_G is the natural equivalence $\Psi_{BG,[\xi_G]}$: $\varphi_G \to H(-, BG)$ given by $\Psi_{BG,[\xi_G]}(X)([\xi]) = [f]$, such that $f^*\xi_G \cong \xi$.

An Eilenberg–Mac Lane space is characterized by having its homotopy concentrated in one fixed dimension, i.e., if $K(A; j)$ is such a space, then

$$\pi_i K(A; j) = \begin{cases} A, & i = j \\ 0, & i \neq j \end{cases}$$

In these cases there is the bijection $[X, K(A; j)] \cong H^j(X; A)$ where the right-hand side is the jth cohomology group of X with coefficients in the discrete Abelian group A. Then real and complex line bundles, i.e., vector bundles associated with principal S^0- and S^1-bundles, are classified by the Stiefel–Whitney class w_1 and the Chern class c_1, respectively; this is so because $BS^0 \simeq \mathbb{R}P^\infty = K(\mathbf{Z}_2, 1)$ and $BS^1 \simeq \mathbb{C}P^\infty = K(\mathbf{Z}, 2)$ and there exist bijections w_1: $\mathcal{B}_{S^0}(X) \to H^1(X; \mathbf{Z}_2)$ and c_1: $\mathcal{B}_{S^1}(X) \to H^2(X; \mathbf{Z})$. The second case is relevant for quantum mechanics (see Section 2.3).

Let $\xi = P(X, G)$ be a G-bundle, p in P, $x = \pi(p)$ in X its projection, and $P_x = \pi^{-1}(\{x\})$ the fiber over x. The vertical space at p is the tangent space to P_x at p: $V_p = T_p P_x$. A connection H in ξ is a smooth assignment of a vector space H_p at each p satisfying the following two conditions: (i) H_p is isomorphic to $T_x X$ through $\pi_{*p}|_{H_p}$ (the restriction to H_p of the differential of π at p) and (ii) $H_{\psi_g(p)} = \psi_{g*p}(H_p)$ for each p in P and g in G. As a consequence $T_p P$, the tangent space to P at p, splits into the direct sum $H_p \oplus V_p$; H_p is called the horizontal space at p. A connection is determined by a smooth 1-form ω on P with values in $\mathcal{G}$, by defining $H_p := \ker \omega_p$.

A universal connection on ξ_G is a family of connections $\{\omega_0\}_{r=1}^{\infty}$, where each ω_r is a connection on ξ_G^r, such that if ω is a connection on ξ: $G \to P \xrightarrow{\pi} X$, then, for some r, there exists a smooth map f: $X \to B_r G$ such that $\omega = \hat{f}^*(\omega_r)$, where $\hat{f}$: $P \to P_r G$ satisfies $\pi_r \circ \hat{f} = f \circ \pi$. Narasimhan and Ramanan (1961) proved that if G is a compact Lie group, then universal connections exist.

Given a smooth curve c in X through the point x, there is a unique curve $c^\uparrow$ in P through p in P_x (the lifting of c by H through p) with horizontal velocity vector $\dot{c}^\uparrow$ at each point. Thus for each path and connection there is a canonical diffeomorphism P_c^H: $P_x \to P_{x'}$ called parallel transport. (This fact is precisely the reason for the name "connection": the horizontal distribution of vector spaces allows us to identify arbitrary "distant" fibers in the total space along prescribed paths in the base space.) If c is a loop at x, then P_c^H is called the holonomy of H at x along c.

Let $G = G^s \subset k(x)$ be a group of matrices ($s = dim_{\mathbb{R}} G$ and $k = \mathbb{R}$ or $\mathbb{C}$), ω the 1-form of the connection H, c: $[t_0, t_1] \to X$ a loop at x, and β: $[t_0, t_1] \to P$ an arbitrary (auxiliary) lifting of c through $q = \beta(t_0) \in P_x$. Then the (unique) curve g: $[t_0, t_1] \to G$ which gives the horizontal lifting of c through $p \in P_x$, i.e., which is such that $c^{\uparrow}(t) = \beta(t)g(t)$ with $qg(t_0) = p = c^{\uparrow}(t_0)$ satisfies the differential equation (Isham, 1989)

$$\frac{D}{dt}\, g(t) = 0 \tag{2.3}$$

with the covariant derivative operator given by

$$\frac{D}{dt} = \frac{d}{dt} + \omega_{\beta(t)}(\dot{\beta}(t)): \quad G^s \to k(s) \tag{2.4}$$

Thus g is covariantly constant. If the auxiliary curve β passes through p, i.e., if $q = p$, then $g(t_0) = 1$. The iterative solution to (2.3) is given by the "time-ordered" exponential

$$\begin{aligned} g(t)g(t_0)^{-1} &= T \exp - \int_{t_0}^{t} dt'\ \omega_{\beta(t')}(\dot{\beta}(t')) \\ &= 1 + \sum_{m=1}^{\infty} (-1)^m \int_{t_0}^{t} d\tau_1\ \omega_{\beta}(\tau_1)(\dot{\beta}(\tau_1)) \\ &\quad \times \int_{t_0}^{\tau_1} d\tau_2\ \omega_{\beta(\tau_2)}(\dot{\beta}(\tau_2)) \cdots \int_{t_0}^{\tau_{m-2}} d\tau_{m-1}\ \omega_{\beta(\tau_{m-1})}\,(\dot{\beta}(\tau_{m-1})) \\ &\quad \times \int_{t_0}^{\tau_{m-1}} d\tau_m\ \omega_{\beta(\tau_m)}(\dot{\beta}(\tau_m)) \end{aligned} \tag{2.5}$$

Formally this quantity is the limit of a time-ordered product of Lie group exponentials with "infinitesimal" arguments, i.e., infinitesimal vectors in the Lie algebra:

$$\lim_{\epsilon \to 0} \prod_{t_0 \le t' \le t_1} \exp[-\omega_{\beta}(t')(\dot{\beta}(t'))\epsilon]$$

(factors are ordered from right to left with increasing time). We emphasize that (2.5) is a global formula. For each $p \in P$ the set of elements $a \in G$ such that $c^{\uparrow}(t_1) = c^{\uparrow}(t_0)a$ for c in the loop space $\Omega(X, x)$ of X at $x = \pi(p)$ is a subgroup of G called the holonomy group of H at p. Given c and p, a is determined by $pa = \beta(t_1)g(t_1)$.

If c is contained in the open set U corresponding to the local trivialization (U, φ) of ξ and $\beta(t) = \sigma(c(t))$, where σ is the local section $\sigma(x) = \varphi(x, 1)$, then $c^{\uparrow}(t)$ is given by the (local) formula

$$c^{\uparrow}(t) = \sigma(c(t))\left[T \exp - \int_{t_0}^{t} dt'\ \mathbf{A}(t')\cdot\dot{\mathbf{x}}(t')\right] g(t_0)$$

where the "gauge potential" $\mathbf{A} = \Sigma_{i=1}^{n} A_i\, dx^i$ is the pullback of ω by σ, i.e., $\mathbf{A} = \sigma^*(\omega|_{\pi^{-1}(U)})$ and x^i, $i = 1, \ldots, n = \dim_{\mathbb{R}} X$ are local coordinates on U.

2.3. Projective Spaces

In the framework of category theory the trivial vector space $\mathbb{R}^0$ can be obtained from the empty set ϕ as the identity morphism id_ϕ; since the only metric on $\mathbb{R}^0$ is $g = 0$, the application of the Clifford functor gives the real numbers $Cl_{\mathbb{R}^0,0} \cong \mathbb{R}$. The complex numbers and the quaternions are respectively obtained through the complexification and quaternization of $\mathbb{R}$, which are also functorial, as shown in Section 2.1. The unitary elements $\mathbb{R}_1$, $\mathbb{C}_1$, and $\mathbb{H}_1$ are, respectively, the 0-, 1-, and 3-spheres (the only spheres which are groups):

$$\mathbb{R} \supset \mathbb{R}_1 = \{1, -1\} = S^0 = O(1) \cong \mathbf{Z}_2$$

$$\mathbb{C} \supset \mathbb{C}_1 = \{z \mid |z| = 1\} = S^1 = U(1) \cong SO(2)$$

$$\mathbb{H} \supset \mathbb{H}_1 = \{q \mid |q| = 1\} = S^3 = Sp(1) \cong SU(2)$$

The universal objects (G-principal bundles) naturally associated with these groups are the infinite spheres over the real, complex, and quaternionic projective spaces, respectively,

$$\xi_{\mathbb{R}}^{\infty}: \quad S^0 \to S^\infty \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}P^\infty$$

$$\xi_{\mathbb{C}}^{\infty}: \quad S^1 \to S^\infty \xrightarrow{\pi_{\mathbb{C}}} \mathbb{C}P^\infty$$

$$\xi_{\mathbb{H}}^{\infty}: \quad S^3 \to S^\infty \xrightarrow{\pi_{\mathbb{H}}} \mathbb{H}P^\infty$$

where for the real case

$$S^\infty = \bigcup_{i=1}^{\infty} S^{i-1} \subset \bigcup_{i=1}^{\infty} \mathbb{R}^i = \mathbb{R}^\infty, \qquad S^{i-1} \subset \mathbb{R}^i, \qquad \mathbb{R}P^\infty = \bigcup_{i=0}^{\infty} \mathbb{R}P^i$$

for the complex case

$$S^\infty = \bigcup_{i=1}^{\infty} S^{2i-1} \subset \bigcup_{i=1}^{\infty} \mathbb{C}^i = \mathbb{C}^\infty, \qquad S^{2i-1} \subset \mathbb{C}^i, \qquad \mathbb{C}P^\infty = \bigcup_{i=0}^{\infty} \mathbb{C}P^i$$

and for the quaternionic case

$$S^\infty = \bigcup_{i=1}^{\infty} S^{4i-1} \subset \bigcup_{i=1}^{\infty} \mathbb{H}^i = \mathbb{H}^\infty, \qquad S^{4i-1} \subset \mathbb{H}^i, \qquad \mathbb{H}P^\infty = \bigcup_{i=0}^{\infty} \mathbb{H}P^i$$

Notice that we are identifying k^n as a subspace of k^{n+1} for $k = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, and also for spheres and projective spaces. In all cases the infinite spaces are given the inductive (also called finite) topology; then the vector spaces K^∞ ($K = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$), which have the standard strictly positive inner products, are not Hilbert spaces since they are not complete; moreover, they are not even pre-Hilbert spaces, since the inductive topology is not metrizable (Willard, 1970).

There are infinite sequences of nontrivial (except for the first bundle in each sequence) bundles (Hopf bundles) and bundle morphisms (inclusions) which for the real, complex, and quaternionic cases are shown in the following diagrams:

$$
\begin{array}{ccccccccccccc}
S^0 & & S^0 & & S^0 & & S^0 & & & S^0 & & & S^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
S^0 & \xrightarrow{i_0} & S^1 & \xrightarrow{i_1} & S^2 & \xrightarrow{i_2} & S^3 & \xrightarrow{i_3} & \cdots \to & S^{i-1} & \xrightarrow{i_{i-1}} & \cdots \to & S^\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
\{\text{pt.}\} & \xrightarrow{j_0} & \mathbb{R}P^1 & \xrightarrow{j_1} & \mathbb{R}P^2 & \xrightarrow{j_2} & \mathbb{R}P^3 & \xrightarrow{j_3} & \cdots \to & \mathbb{R}P^{i-1} & \xrightarrow{j_{i-1}} & \cdots \to & \mathbb{R}P^\infty
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
S^1 & & S^1 & & S^1 & & S^1 & & & S^1 & & & S^1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
S^1 & \xrightarrow{k_1} & S^3 & \xrightarrow{k_3} & S^5 & \xrightarrow{k_5} & S^7 & \xrightarrow{k_7} & \cdots \to & S^{2i-1} & \xrightarrow{k_{2i-1}} & \cdots \to & S^\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
\{\text{pt.}\} & \xrightarrow{l_0} & \mathbb{C}P^1 & \xrightarrow{l_1} & \mathbb{C}P^2 & \xrightarrow{l_2} & \mathbb{C}P^3 & \xrightarrow{l_3} & \cdots \to & \mathbb{C}P^{i-1} & \xrightarrow{l_{i-1}} & \cdots \to & \mathbb{C}P^\infty
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
S^3 & & S^3 & & S^3 & & S^3 & & & S^3 & & & S^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
S^3 & \xrightarrow{m_3} & S^7 & \xrightarrow{m_7} & S^{11} & \xrightarrow{m_{11}} & S^{15} & \xrightarrow{m_{15}} & \cdots \to & S^{4i-1} & \xrightarrow{m_{4i-1}} & \cdots \to & S^\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
\{\text{pt.}\} & \xrightarrow{n_0} & \mathbb{H}P^1 & \xrightarrow{n_1} & \mathbb{H}P^2 & \xrightarrow{n_2} & \mathbb{H}P^3 & \xrightarrow{n_3} & \cdots \to & \mathbb{H}P^{i-1} & \xrightarrow{n_{i-1}} & \cdots \to & \mathbb{H}P^\infty
\end{array}
$$

($\mathbb{R}P^1 \cong S^1$, $\mathbb{C}P^1 \cong S^2$, and $\mathbb{H}P^1 \cong S^4$). The sequences are natural in the sense that each sequence is contained in its infinite limit, which exists by functoriality; in particular, the projective spaces $\mathbb{C}P^{n-1}$ which correspond to the physical states of quantum systems with a finite-dimensional Hilbert space $\mathbb{C}^n$ are also natural. Notice that finite spheres (S^{2n-1}) which are noncontractible are contained in Hilbert spaces ($\mathbb{C}^n$), while for the infinite sphere the opposite occurs: S^∞ is contractible, but $\mathbb{C}^\infty$ is not Hilbert.

There are natural relationships between these sequences, given by the *projective twistor bundle* (Ward and Wells, 1990) τ: $S^2 - \mathbb{C}P^3 \xrightarrow{p} \mathbb{H}P^1$ in the complex-quaternionic case and the principal bundle $S^1 \rightarrow \mathbb{R}P^3 \xrightarrow{q} \mathbb{C}P^1$ in the real-complex case; in fact, if $[\vec{z}] = \vec{z}\mathbb{C}^*$ with $\vec{z} = (z_1, z_2, z_3, z_4) \in \mathbb{C}^{4*}$ and $[\vec{x}] = \vec{x}\mathbb{R}^*$ with $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^{4*}$ are points in $\mathbb{C}P^3$ and $\mathbb{R}P^3$, then $p[\vec{z}] = (z_1 + jz_2, z_3 + jz_4)\mathbb{H}^*$ and $q[\vec{x}] = (x_1 + ix_2, x_3 + ix_4)\mathbb{C}^*$ are corresponding projections onto $\mathbb{H}P^1$ and $\mathbb{C}P^1$, while the compositions $j_2 \circ j_1$ and $l_2 \circ l_1$ are inclusions of the spheres S^1 and S^2 into $\mathbb{R}P^3$ and $\mathbb{C}P^3$, respectively; notice that τ is not a principal bundle, since S^2 is not a group; however since $\mathbb{H}P^1 = P(\mathbb{H}^2)$, $\mathbb{H} \cong \mathbb{C}^2$, and $P(\mathbb{C}^2) = \mathbb{C}P^1$, then τ is a fiber bundle with fiber S^2.

Using the bijections mentioned in Section 2.2, one obtains the classification of all the principal $O(1)$-bundles in the real case and principal $U(1)$-bundles in the complex case:

$$w_1\colon\ \mathcal{B}_{S^0}(\mathbb{R}P^{i-1}) \to H^1(\mathbb{R}P^{i-1}; \mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}_2, & i \geq 2 \\ 0, & i = 1 \end{cases}$$

means that in the real case for each bundle in the sequence (except for the first one) one has the additional trivial bundle $S^0 \to \mathbb{R}P^{i-1} \times S^0 \to \mathbb{R}P^{i-1}$ (two copies of $\mathbb{R}P^{i-1}$), while

$$c_1\colon\ \mathcal{B}_{S^1}(\mathbb{C}P^{l-1}) \to H^2(\mathbb{C}P^{l-1}; \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & l \geq 2 \\ 0, & l = 1 \end{cases}$$

means that in the complex case for each bundle in the sequence (except for the first one) one has an additional infinite set of bundles $S^1 \to P_k^{2l-1} \to \mathbb{C}P^{l-1}$ with $k \in \mathbf{Z}$, $k \neq 1$ ($P_1^{2l-1} = S^{2l-1}$ and $k = 0$, all l correspond to trivial bundles). The *monopole bundles* corresponding to the case $l = 2$, which with proper connections represent Dirac magnetic monopoles of charge k, include the Hopf bundle of spheres $S^1 \to S^3 \to S^2$ for $k = 1$ (Wu and Yang, 1975), which in addition in quantum mechanics represents a general 2-state system (in particular the spin-1/2 system), and the bundle $S^1 \to \mathbb{R}P^3 \xrightarrow{q} \mathbb{C}P^1$ for $k = 2$ since $P_2^3 \cong SO(3) \cong \mathbb{R}P^3$.

Since $\mathbb{H}P^\infty$ is not an Eilenberg–Mac Lane space, for the principal $SU(2)$-bundles we do not have a classification as that given above for the real and complex cases; the long exact homotopy sequence (Steenrod, 1951), however, applied to the universal bundle for $G = S^3$ leads to $\mathcal{B}_{S^3}(S^4) \cong \mathbf{Z}$ [similarly as in the complex case, in which $\mathcal{B}_{S^1}(S^2) \cong \mathbf{Z}$], which are the well-known *instanton bundles* of t'Hooft and Polyakov; in particular, the Hopf bundle of spheres $S^3 \to S^7 \to S^4$ represents the unit of instanton charge (Trautman, 1977).

3. QUANTUM EVOLUTION

In this section we shall review the relation between the Schrödinger equation for the wave function ψ, the Schrödinger–von Neumann equation for the density matrix ρ (in the present case for a pure state), and the Aharonov–Anandan equation for the geometric part of the wave function $\tilde{\psi}$. Though these results are well known (Bohm, 1993a), we include them here for completeness and to emphasize some deep but natural geometrical aspects of quantum mechanics. The results are valid for both finite ($\mathbb{C}^n$)- and infinite ($\mathcal{H}$)-dimensional Hilbert spaces; as we saw in Section 2.3, however, the infinite-dimensional case does not correspond to the topological limit $n \to \infty$ of the finite cases, since $\mathbb{C}^\infty$ is not a Hilbert space; the only role of this limit is to codify in a single bundle all the information contained in the finite-dimensional cases. The geometry of the infinite-dimensional projective Hilbert bundle $S^1 \to S(\mathcal{H}) \xrightarrow{\pi} \mathcal{P}(\mathcal{H})$ and the corresponding universal connection will be discussed in the next section.

A (pure) *physical state* in quantum mechanics is represented by a *ray* in the Hilbert space V ($\mathbb{C}^n$ or $\mathcal{H}$) of the system, i.e., by a complex line through the origin in this space, and so it consists of a point in the projective space $\mathcal{P}(V)$. This point has a *density matrix* representation through the complex-linear operator (in Dirac notation) $\rho = |\psi\rangle \otimes_{\mathbb{C}} \langle\psi| \in V \otimes_{\mathbb{C}} V^* \cong Hom_{\mathbb{C}}(V, V)$, the space of complex-linear maps $V \to V$, and where V^* is the dual of V, $|\psi\rangle$ is a normalized ($\langle\psi|\psi\rangle = 1$) state vector to account for the probabilistic interpretation, and phase invariance symmetry, i.e., the absence of physical consequences of global multiplication of the wave function ψ by elements of $U(1)$, is automatically incorporated since the Hermitian inner product in V requires that $\langle\psi| \to e^{-i\alpha}\langle\psi|$ if $|\psi\rangle \to e^{i\alpha}|\psi\rangle$; ρ is normally denoted by $|\psi\rangle\langle\psi|$. Also, $\rho^2 = \rho$, i.e., ρ is a projection operator: if $|\varphi\rangle \in V$, then $\rho(|\varphi\rangle) = c_\varphi|\varphi\rangle$ with $c_\varphi = \langle\psi|\varphi\rangle \in \mathbb{C}$; if $|\varphi\rangle$ is normalized, then $|c_\varphi|^2$ is the probability to find the system described by the state vector $|\varphi\rangle$ in the state $|\psi\rangle$, and this probability does not change if $|\varphi\rangle$ is replaced by $e^{i\alpha}|\varphi\rangle$. Thus we have the identifications

$$\{\text{pure physical states}\} \leftrightarrow \mathcal{P}(V) \leftrightarrow \{|\psi\rangle \otimes_{\mathbb{C}} \langle\psi|,\ |\psi\rangle \in V, \langle\psi|\psi\rangle = 1\}$$

If H is the Hamiltonian operator, then the normalized state vector $|\psi\rangle$ satisfies the Schrödinger equation $i\hbar(\partial/\partial t)|\psi\rangle = H|\psi\rangle$ and because of the hermiticity of H the vector $\langle\psi|$ of the dual space obeys $i\hbar(\partial/\partial t)\langle\psi| = -\langle\psi|H$. Then the density matrix for the pure state satisfies the Schrödinger–von Neumann equation

$$i\hbar \frac{\partial}{\partial t}\rho = [H, \rho]$$

with $[H, \rho] = H\rho - \rho H$. (The density matrix for a mixed state $\rho = \Sigma_i w_i|\psi_i\rangle\langle\psi_i|$ obeys the same equation, here the w_i are probabilities and satisfy $\Sigma_i w_i = 1$.) A solution $\rho(t)$ to this equation with a given initial condition $\rho(0)$ is a path in $\mathcal{P}(V)$, and a corresponding solution $\psi(t)$ to the Schrödinger equation is a path in the unit sphere $S(V)$ which projects onto $\rho(t)$: $\pi(\psi(t)) = \rho(t)$, i.e., $\psi(t)$ is a lifting of $\rho(t)$. However, since

$$\langle\psi|\dot{\psi}\rangle = \langle\psi|\left(\frac{\partial}{\partial t}|\psi\rangle\right) = -\frac{i}{\hbar}\langle\psi|H|\psi\rangle$$

is in general different from zero, $|\psi\rangle$ is not a horizontal lifting, which, as we saw in Section 2.2, requires a horizontal velocity vector $|\dot{\psi}\rangle$ at each point. [$|\psi(t)\rangle$ is an element of the fiber at $\rho(t)$, and horizontality is defined by the canonical or the universal connection.] Since $\dot{\psi} \in T_\psi S(V)$, then, as it should be, $\mathbb{R}e\langle\psi|\dot{\psi}\rangle = 0$, i.e., in the real sense (but not in the complex one) ψ and $\dot{\psi}$ are orthogonal.

The horizontal lifting of $\rho(t)$ in $S(V)$ is the Aharonov–Anandan wave function defined by

$$\tilde{\psi}(t) := \left\{\exp\left[\frac{i}{\hbar}\int_0^t dt'\ \langle\psi(t')|H(t')|\psi(t')\rangle\right]\right\}\psi(t)$$

which obeys the equation $i\hbar(\partial/\partial t)\tilde{\psi} = \tilde{H}(t)\tilde{\psi}(t)$ with $\tilde{H}(t) = H(t) - \langle\psi(t)|H(t)|\psi(t)\rangle$ and the initial condition $\tilde{\psi}(0) = \psi(0)$; as a consequence, $\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle = 0$ [also $\langle\psi(t)|\dot{\tilde{\psi}}(t)\rangle = 0$]. $\tilde{\psi}(t)$ can be considered as the *geometrical part of* $\psi(t)$ since according to its definition $\tilde{\psi}$ is obtained from ψ by multiplying by the inverse of the dynamical phase factor $\exp(i\alpha_{\text{dyn}})$, where

$$\alpha_{\text{dyn}} = -\frac{1}{\hbar}\int_0^t dt'\ \langle\psi(t')|H(t')|\psi(t')\rangle$$

So, *the time evolution of the Aharonov–Anandan wave function is given by the parallel transport determined by the canonical (or universal) connection on the Stiefel (or universal) bundle along the solution of the Schrödinger–von Neumann equation in the projective space* (Bohm *et al.*, 1993). For a closed path in this space the holonomy in $S(V)$ is observable and it gives the Berry–Simon–Aharonov–Anandan phase (the geometric phase) which was originally discovered by Berry for adiabatic processes.

Since the Schrödinger–von Neumann equation is a dynamical equation, we cannot argue that the quantum evolution process has been completely understood in geometrical terms. However, recent developments on the geometrization of quantum mechanics (Ashtekar and Schilling, 1994; Corichi and Ryan, 1995) could in principle be applied to that equation, hopefully leading to a complete geometrical picture. This is the subject of further research.

Also, using the Feynman path integral formulation of quantum mechanics in terms of the density matrix, Ajanapon (1987, 1988) showed that the classical limit of pure states, which consists in maintaining only the diagonal elements of the density matrix and neglecting the off-diagonal ones, is the Liouville equation of classical statistical mechanics (the deterministic classical limit consisting of Newton or Hamilton equations requires that one start out initially from a mixed quantum state). The discussion above relating the density matrix operators with the space of physical states suggests that the Liouville equation is the correct *physical* classical limit of pure quantum states. It would then be of interest to study this relation between the Schrödinger–von Neumann equation and the Liouville equation from a bundle-theoretic (i.e., geometric) point of view.

4. PROJECTIVE HILBERT BUNDLE

In this last section we shall describe in detail the differentiable structure of the classifying bundle ξ_C: $S^1 \to S(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ associated with an infinite-dimensional Hilbert space, and of its universal connection, responsible for the geometric phases. First, for completeness, we shall present some details of our proof (Aguilar and Socolovsky, 1996) that already at the topological level ξ_C is a classifying bundle for S^1, which in particular gives the explicit homotopy equivalence between CP^∞ and $\mathcal{P}(\mathcal{H})$. The physical importance of these results consists in that they provide a natural "bridge" between the description of quantum systems with a finite-dimensional Hilbert space and that of quantum systems with an infinite-dimensional Hilbert space; clearly the *rationale* for this unity is provided by the concept of homotopy.

Let $E = \lim_{V \subset E} V$ as in Section 2.3, i.e., E is a real or complex vector space given as a limit of its finite-dimensional subspaces (ordered by inclusion); this topology is called the finite or inductive topology. Then the following theorem holds (Palais, 1966): E is a topological vector space (i.e., the sum of vectors and the product of vectors by scalars are continuous functions) if and only if E has a countable (algebraic) basis. Real or complex *infinite-dimensional Hilbert spaces* $\mathcal{H}$, however, have an uncountable basis and so they do not have the finite topology, but they are *complete inner product linear spaces*. The norm induced by the inner product, $\|v\| := \sqrt{(v, v)}$, makes them metric spaces with a translation-invariant distance and therefore they are *locally convex* and *completely normal* topological vector spaces, which respectively means that the topology has a basis consisting of convex sets (the open balls) and that disjoint closed sets can be separated by continuous functions [if A and B are disjoint closed sets in the metric space (E, d), then

$$f(v) := \frac{d(v, A)}{d(v, A) + d(v, B)} \in [0, 1]$$

satisfies $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$, with $d(v, C) = \inf_{c \in C} d(v, c)$ for any $C \subset E$]; in particular, A or B can be replaced by a point and therefore Hilbert spaces are *completely regular*; then they are also *Hausdorff* spaces.

Notice that the decomposition of the inner product in a complex Hilbert space $\mathcal{H}$ into its real and imaginary parts, $(v, w) = \mathbb{R}e(v, w) + i\, Im(v, w)$, makes it possible to consider $\mathcal{H}$ as a complete real vector space with a bilinear symmetric positive-definite inner product given by $\mathbb{R}e(v, w)$; the convergence of Cauchy sequences is a consequence of the equality of distances $d_{\mathbb{R}}(v, w) = d_{\mathbb{C}}(v, w)$. The imaginary part is a symplectic inner product.

Any (infinite-dimensional) Hilbert space $\mathcal{H}$ (Conway, 1990) has a topological basis or *complete orthonormal system* $\epsilon = \{e_\alpha\}_{\alpha \in J}$ with $(e_\alpha, e_\beta) = 0$ if $\alpha \neq \beta$ and $(e_\alpha, e_\alpha) = 1$ for all α, in terms of which each vector belonging to $\mathcal{H}$ is given by its *Fourier series* $v = \Sigma_{\alpha \in J} (v, e_\alpha) e_\alpha$ (only a countable subset of the indexing set J contributes to the sum).

If ϵ is countable, then $\mathcal{H}$ is called *separable*, otherwise it is *nonseparable*. The *canonical separable* complex Hilbert space is

$$l^2 := \left\{ \vec{z} \equiv (z_1, z_2, \ldots), z_i \in \mathbb{C}, \sum_{i=1}^{\infty} |z_i|^2 < \infty \right\}$$

If $e_1 = (1, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, $\ldots$ is the standard topological basis then the Fourier series for $\vec{z}$ is $\vec{z} = \Sigma_{i=1}^{\infty} z_i e_i$, and clearly $\mathbb{C}^\infty \cong span(\{e_i\}_{i=1}^{\infty})$, i.e., the topological basis of l^2 is the algebraic basis of $\mathbb{C}^\infty$ with $l^2 = closure(\mathbb{C}^\infty)$; in particular, this implies that $\mathbb{C}^\infty$ is not a closed subset of l^2. l^2 can be identified with the set of functions

$$l^2(\mathbb{N}) := \left\{ \varphi: \mathbb{N} \to \mathbb{C}, n \mapsto \varphi(n) \equiv \varphi_n \text{ with } \sum_{n=1}^{\infty} |\varphi_n|^2 < \infty \right\}$$

Then if $\mathcal{H}$ is an arbitrary separable Hilbert space and $\epsilon = \{e_i\}_{i=1}^{\infty}$ is a topological basis, it can be shown that the continuous function

$$\psi: \mathcal{H} \to l^2(\mathbb{N}), v \mapsto \psi(v): \mathbb{N} \to \mathbb{C}, n \mapsto \psi(v)(n) := (v, e_n)$$

is a Hilbert space isomorphism. In particular, ψ preserves the inner product:

if $v, w \in \mathcal{H}$, then $v = \Sigma_{k=1}^{\infty} v_k e_k$ and $w = \Sigma_{k=1}^{\infty} w_k e_k$ with $v_k = (v, e_k)$ and $w_k = (w, e_k)$, then

$$(v, w) = \sum_{k=1}^{\infty} v_k \overline{w}_k = \sum_{k=1}^{\infty} (v, e_k)\overline{(w, e_k)}$$

$$= \sum_{k=1}^{\infty} \psi(v)(k)\overline{\psi(w)(k)} = (\psi(v), \psi(w)) \qquad \text{in} \quad l^2(\mathbb{N})$$

Similarly, for an arbitrary infinite-dimensional Hilbert space $\mathcal{H}$ with basis ϵ one defines the space of functions

$$l^2(\epsilon) := \{\varphi: \epsilon \to \mathbb{C},\ e_\alpha \mapsto \varphi(e_\alpha) \equiv \varphi_\alpha \text{ with } \sum_{\alpha \in J} |\varphi_\alpha|^2 < \infty\}$$

and

$$\psi: \quad \mathcal{H} \to l^2(\epsilon), \quad v \mapsto \psi(v): \quad \epsilon \to \mathbb{C},\ e_\alpha \mapsto \psi(v)(e_\alpha) := (v, e_\alpha)$$

is again an isomorphism. Since ϵ can be canonically identified with its set of indices, this gives the well-known result that all Hilbert spaces with the same cardinality are isomorphic to each other; notice that in general the isomorphism is not canonical. In the quantum mechanical case, however, if H is the Hamiltonian of the system, then the set of its orthonormalized eigenstates $\epsilon_0 = \{\varphi_\lambda\}_{\lambda \in J}$ is a natural topological basis of the Hilbert space, $\mathcal{H} = closure(span(\epsilon_0))$, and in the particular case that $J = \mathbb{N}$ one has the canonical isomorphism $\Phi: \mathcal{H} \to l^2(\mathbb{N})$, $\psi \mapsto \Phi(\psi): \mathbb{N} \to \mathbb{C}$, $n \mapsto (\psi, \varphi_n)$.

By the Stone theorem (Willard, 1970) Hilbert spaces are paracompact and therefore admit continuous partitions of unity. The *Hilbert sphere*

$$S(\mathcal{H}) := \{v \in \mathcal{H} \mid \|v\| = 1\}$$

being a metric subspace of $\mathcal{H}$, is *completely normal* and *paracompact, but not compact.* One has the continuous action $S(\mathcal{H}) \times S^1 \to S(\mathcal{H})$, $(v, z) \mapsto vz$, and the projection $q: S(\mathcal{H}) \to S(\mathcal{H})/S^1$, $q(v) = [v] = \{vz\}_{z \in S^1}$, onto the quotient or orbit space, $\mathcal{P}(\mathcal{H}) := S(\mathcal{H})/S^1$: the *projective Hilbert space*; since $S(\mathcal{H})$ and S^1 are Hausdorff spaces and S^1 is compact, then $\mathcal{P}(\mathcal{H})$ is *Hausdorff*, and q is open, since it is the projection associated with an action, and because of the compactness of S^1 it is also closed; then $\mathcal{P}(\mathcal{H}) = q(S(\mathcal{H}))$ is *paracompact* since images of paracompact spaces by closed functions are paracompact.

Let V be a locally convex topological vector space and E a dense vector subspace with countable algebraic dimension and equipped with the finite topology; let Θ be an open subset of V and $\tilde{\Theta} := \Theta \cap E$. By a theorem of Palais (1966) if V is metrizable, then the inclusion $\tilde{\Theta} \xrightarrow{j} \Theta$ is a homotopy equivalence (notice that E is not a topological subspace of V). We apply this result to $V = \mathcal{H}$, an arbitrary *separable* Hilbert space, $E = \mathbb{C}^\infty$, $\Theta = \mathcal{H} - \{0\}$, and $\tilde{\Theta} \simeq (\mathcal{H} - \{0\}) \cap \mathbb{C}^\infty = \mathbb{C}^\infty - \{0\}$; the inclusion j is the restriction to $\mathbb{C}^\infty - \{0\}$ of the inclusion k given by the diagram

$$\begin{array}{ccc} & \mathcal{H} & \\ {}^{k}\nearrow & & \nwarrow^{\psi^{-1}} \\ \mathbb{C}^\infty & \xrightarrow{i} & l^2 \end{array}$$

i.e., $k = \psi^{-1} \circ i$ and $j = k|_{\mathbb{C}^\infty - \{0\}}$, where ψ: $\mathcal{H} \to l^2(\mathbb{N})$ is an isomorphism which depends on the basis ϵ of $\mathcal{H}$ and $i(z_1, \ldots, z_r) = (z_1, \ldots, z_r, 0, \ldots)$; then $\mathbb{C}^\infty - \{0\} \xrightarrow{j} \mathcal{H} - \{0\}$ is a homotopy equivalence. On the other hand, the continuous functions $S(\mathcal{H}) \xrightarrow{\iota} \mathcal{H} - \{0\}$ given by the inclusion and $\mathcal{H} - \{0\} \xrightarrow{r} S(\mathcal{H})$ given by $r(v) = v/\|v\|$ satisfy $r \circ \iota = id_{S(\mathcal{H})}$ and $\iota \circ r \sim id_{\mathcal{H}-\{0\}}$ with the homotopy given by H: $(\mathcal{H} - \{0\}) \times I \to \mathcal{H} - \{0\}$, $H(v, t) = (1 - t)v + tv/\|v\|$ [so $H(v, 0) = H_0(v) = v$, i.e., $H_0 = id_{\mathcal{H}-\{0\}}$ and $H(v, 1) = H_1(v) = v/\|v\|$, i.e., $H_1 = \iota \circ r$]; then ι is a homotopy equivalence and therefore $S(\mathcal{H}) \simeq \mathcal{H} - \{0\}$. One has the diagram

$$\begin{array}{ccc} \mathbb{C}^\infty - \{0\} & \xrightarrow{j} & \mathcal{H} - \{0\} \\ {\scriptstyle \iota_0}\uparrow & & {\scriptstyle \iota}\uparrow\downarrow{\scriptstyle r} \\ S^\infty & \xrightarrow{\alpha} & S(\mathcal{H}) \end{array}$$

where $\alpha := r \circ j \circ \iota_0$, being the composition of homotopy equivalences (the canonical inclusion ι_0 is also a homotopy equivalence), is a homotopy equivalence; explicitly, if $\vec{z} \in S^\infty$, then $\vec{z} = (z_1, \ldots, z_s)$ with $\sum_{i=1}^s |z_i|^2 = 1$ for some s, and

$$\begin{aligned} \alpha(\vec{z}) &= r \circ j \circ \iota_0(\vec{z}) = r(j(\vec{z})) = r(\psi^{-1} \circ i(\vec{z})) = r(\psi^{-1}(z_1, \ldots, z_s, 0, \ldots)) \\ &= \psi^{-1}(z_1, \ldots, z_s, 0, \ldots) = \sum_{k=1}^{s} z_k e_k \end{aligned}$$

i.e., α is an inclusion of S^∞ into $S(\mathcal{H})$; obviously, α is an S^1-map (S^1-equivariant), i.e., $\alpha(\vec{z}z) = \alpha(\vec{z})z$ for $z \in S^1$. Therefore $S(\mathcal{H}) \simeq S^\infty \simeq \{pt.\}$, i.e., $S(\mathcal{H})$ is contractible. In Section 4.6 we shall prove that

$$\xi_{\mathbb{C}}: \quad S^1 \to S(\mathcal{H}) \xrightarrow{q} \mathcal{P}(\mathcal{H})$$

is a smooth principal S^1-bundle, and therefore a topological S^1-bundle. Since any principal bundle on a paracompact space is numerable, then $\xi_{\mathbb{C}}$ is numerable and therefore it is a classifying bundle for S^1. By universality, $\mathbb{C}P^\infty \simeq \mathcal{P}(\mathcal{H})$. The equivariance of α implies that the function $\bar{\alpha}$: $\mathbb{C}P^\infty \to \mathcal{P}(\mathcal{H})$, x

$\mapsto \overline{\alpha}(x) := q(\alpha(y))$ with $y \in \pi_C^{-1}(\{x\})$ is well defined [i.e., if y' is another element of the fiber over x, then $q(\alpha(y')) = q(\alpha(y))$]; then one has the S^1-bundle morphism given by the following diagram:

$$
\begin{array}{ccc}
S^1 & & S^1 \\
\big\downarrow & & \big\downarrow \\
S^\infty & \xrightarrow{\alpha} & S(\mathcal{H}) \\
{\scriptstyle \pi_C}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
CP^\infty & \xrightarrow{\overline{\alpha}} & \mathcal{P}(\mathcal{H})
\end{array}
$$

The important practical result is that $\overline{\alpha}$ explicitly gives the homotopy equivalence between CP^∞ and $\mathcal{P}(\mathcal{H})$. In fact, by a theorem of Dold (1963), if X and X' are respectively free and contractible G-spaces, then (i) there exists a G-map β: $X' \to X$, and (ii) any two such maps are G-homotopic (i.e., the two maps are homotopic and the homotopy between them is a G-map). We apply this result to $X' = S(\mathcal{H})$ and $X = S^\infty$; then there exists an S^1-map β: $S(\mathcal{H}) \to S^\infty$. So $\beta \circ \alpha$: $S^\infty \to S^\infty$ and $\alpha \circ \beta$: $S(\mathcal{H}) \to S(\mathcal{H})$ are S^1-equivariant, and since the identities id_{S^∞} and $id_{S(\mathcal{H})}$ are S^1-maps, then $\beta \circ \alpha \sim id_{S^\infty}$($S^1$-homotopic) and $\alpha \circ \beta \sim id_{(\mathcal{H})}$($S^1$-homotopic), i.e., β is the homotopy inverse of α. This result passes to the quotients and $\overline{\beta}$: $\mathcal{P}(\mathcal{H}) \to CP^\infty$ is the homotopy inverse of $\overline{\alpha}$. Therefore $\overline{\alpha}([(z_1, z_2, \ldots)]) = [\Sigma\, z_k e_k]$ is a homotopy equivalence.

If in the above constructions $\mathcal{H} = l^2$, then $\psi = id_{l^2}, j = \iota|_{C^\infty - \{0\}} \equiv j_0$ and

$$\alpha(z_1, \ldots, z_r) = (z_1, \ldots, z_r, \ldots) \equiv \alpha_0(z_1, \ldots, z_r)$$

The corresponding diagrams are

$$
\begin{array}{ccc}
C^\infty - \{0\} & \xrightarrow{j_0} & l^2 - \{0\} \\
{\scriptstyle \iota_0}\big\downarrow & & {\scriptstyle \iota}\big\uparrow\big\downarrow{\scriptstyle r} \\
S^\infty & \xrightarrow{\alpha_0} & S(l^2)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
S^1 & & S^1 \\
\big\downarrow & & \big\downarrow \\
S^\infty & \xrightarrow{\alpha_0} & S(l^2) \\
{\scriptstyle \pi_C}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
CP^\infty & \xrightarrow{\overline{\alpha}_0} & \mathcal{P}(l^2)
\end{array}
$$

In the following we shall discuss the differentiable structure of the classifying bundle ξ_C and the universal connection on it. For simplicity we shall restrict ourselves to separable Hilbert spaces, and in particular to the case $\mathcal{H} = l^2(\mathbb{N})$.

4.1. $S(\mathcal{H})$ as an Infinite-Dimensional Real Banach Manifold (Choquet-Bruhat *et al.*, 1982; Abraham *et al.*, 1988)

Let $f: \mathcal{H} \to \mathbb{R}$ be given by $f(x) := \|x\|^2$; then $S(\mathcal{H}) = f^{-1}(\{1\})$. We shall prove that $1 \in \mathbb{R}$ is a regular value of f. The derivative of f at x in the direction v (both x and v are in $\mathcal{H}$) is defined by $Df(x)(v) = \lim_{t\to 0}(f(x + tv) - f(x))/t$ and an easy calculation gives $Df(x)(v) = (x, v) + \overline{(x, v)}$; in particular, $Df(x)(x) = 2 \neq 0$ for any $x \in S(\mathcal{H})$, which shows that $Df(x)$ is surjective at these points since $Df(x)$: $T_x\mathcal{H} = \mathcal{H} \to \mathbb{R}$ is (real) linear for any $x \in \mathcal{H}$. Let us now verify that at each $x \in S(\mathcal{H})$, $\ker(Df(x))$ has a *complement*, i.e., there exists a closed real Banach subspace $\mathcal{V}_x$ of $\mathcal{H}$ such that $\ker(Df(x)) \oplus_{\mathbb{R}} \mathcal{V}_x = \mathcal{H}$. In fact, for $x = e_1 = (1, 0, \ldots)$, $Df(e_1)(v) = z_1 + \bar{z}_1$, where $v = (z_1, z_2, \ldots)$, so $\ker(Df(e_1)) = \{v = (i\lambda, z_2, \ldots), \lambda \in \mathbb{R}\}$ and then $\mathcal{V}_{e_1} = \{\rho, 0, \ldots), \rho \in \mathbb{R}\} \cong \mathbb{R}$; notice that $\mathcal{V}_{e_1}$ is the orthogonal complement of $\ker(Df(e_1))$ with respect to the real inner product in $\mathcal{H}$, since $\mathbb{R}e((\rho, 0, \ldots), (i\lambda, z_2, \ldots)) = \mathbb{R}e(-i\lambda\rho) = 0$ for $x \neq e_1$; $\ker(Df(x)) = \{v \in \mathcal{H} \mid \mathbb{R}e(x, v) = 0\}$ and $\mathcal{V}_x = \{w \in \mathcal{H} \mid \mathbb{R}e(w, v) = 0$ for all $v \in \ker(Df(x))\}$, which is closed since the short exact sequence

$$0 \to ker(Df(x)) \to \mathcal{H} \xrightarrow{Df(x)} \mathbb{R} \to 0$$

implies $\mathcal{H} \cong \ker(Df(x)) \oplus_{\mathbb{R}} \mathbb{R}$ and so $\mathcal{V}_x \cong \mathbb{R}$ (the isomorphisms are not canonical). From the implicit function theorem, $S(\mathcal{H})$ is a *closed, regular, real, infinite-dimensional Banach submanifold* of $\mathcal{H}$. In particular, the tangent and normal spaces at $x \in S(\mathcal{H})$ are given by $T_xS(\mathcal{H}) = \ker(Df(x))$ and $N_x = \mathcal{V}_x$, respectively.

4.2. The Canonical Decomposition of $T_{e_1}S(\mathcal{H})$

The tangent space to the Hilbert sphere $S(\mathcal{H})$ at e_1 canonically decomposes into the direct sum $T_{e_1}S(\mathcal{H}) = H_{e_1} \oplus_{\mathbb{R}} V_{e_1}$, where

$$H_{e_1} := \left\{(0, z_2, z_3, \ldots), z_i \in \mathbb{C}, \sum_{i=2}^{\infty} |z_i|^2 < \infty\right\}$$

is the *horizontal space* at e_1, and $V_{e_1} = \{i\lambda, 0, \ldots), \lambda \in \mathbb{R}\}$ is the vertical space, isomorphic to $i\mathbb{R}$, the Lie algebra of S^1. Notice that both H_{e_1} and V_{e_1} are *real* vector spaces. After Section 4.6, H_{e_1} will be the *universal connection* at e_1.

4.3. The Hilbert Sphere as a Homogeneous Space

We shall prove that the Hilbert sphere is diffeomorphic, as a Banach manifold, to the homogeneous space $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$, where $\mathcal{U}(\mathcal{H})$ is the *unitary group* of $\mathcal{H}$ and $\mathcal{U}(\mathcal{H})_{e_1}$ is the isotropy subgroup at e_1.

Let E be a normed vector space and F a Banach space; then the set of linear continuous (bounded) functions (operators) from E to F, $\mathcal{L}(E, F)$, is a Banach space with the operator norm topology: if $A \in \mathcal{L}(E, F)$, then

$$\|A\| = \sup_{x \in E, \|x\|=1} \|A(x)\|$$

A *Banach algebra* $\mathcal{A}$ is a Banach space equipped with an associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ab$, having the usual distributive properties with respect to vector addition and such that $(\lambda a)(\mu b) = (\lambda\mu)(ab)$ and $\|ab\| \leq \|a\| \cdot \|b\|$, where $\lambda, \mu \in K$ and $a, b \in \mathcal{A}$. For example, if E is a Banach space, then $\mathcal{L}(E, E) \equiv \mathcal{L}(E)$ with the product given by the composition of operators is a Banach algebra with unit id_E. If E is real (complex) then $\mathcal{L}(E)$ is a real (complex) algebra. There is the following *theorem*: The set $\mathcal{A}^*$ of invertible elements in a Banach algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ is a *Banach Lie group*, that is, $\mathcal{A}^*$ is a group and a Banach manifold with compatible algebraic and differentiable structures, i.e., the group operations (product and inverse) are smooth. (The idea of the proof is that $\mathcal{A}^*$ is an open set in $\mathcal{A}$, and so it inherits the differentiable structure.) In the example, $\mathcal{L}(E)^* \equiv GL(E)$ is the *general linear group* of E. In particular, for a Hilbert space, $\mathcal{L}(\mathcal{H}) \supset GL(\mathcal{H})$ is a Banach Lie group. Clearly $Lie(GL(\mathcal{H})) = T_{Id}GL(\mathcal{H}) = \mathcal{L}(\mathcal{H})$, which is a real (complex) Banach Lie algebra with $[\alpha, \beta] = \alpha\beta - \beta\alpha$ if $\mathcal{H}$ is a real (complex) Hilbert space.

For each operator A in $\mathcal{L}(\mathcal{H})$ there exists a unique operator A^* called the *adjoint* of A which satisfies $(Ax, y) = (x, A^*y)$ for all $x, y \in \mathcal{H}$; this is a consequence of the canonical isomorphism between $\mathcal{H}$ and its dual space $\mathcal{H}'$. (In the physical literature the usual notation for A^* is $A^\dagger$.) The function $*: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $A \mapsto A^*$, is continuous and real linear [but not complex linear, since $(\lambda A)^* = \bar{\lambda}A^*$]. The sets of operators $\mathcal{L}^s(\mathcal{H}) := \{A \in \mathcal{L}(\mathcal{H}) | A^* = A\}$ (*self-adjoint or Hermitian*) and $\mathcal{L}^a(\mathcal{H}) := \{A \in \mathcal{L}(\mathcal{H}) | A^* = -A\}$ (*skew-* or *anti-Hermitian*) are real Banach closed subspaces of $\mathcal{L}(\mathcal{H})$, and $\mathcal{L}(\mathcal{H}) \to \mathcal{L}^s(\mathcal{H}) \oplus_{\mathbf{R}} \mathcal{L}^a(\mathcal{H})$, $A \mapsto \frac{1}{2}(A + A^*) \oplus_{\mathbf{R}} \frac{1}{2}(A - A^*)$ is an isomorphism.

The *unitary group* of $\mathcal{H}$ is the subgroup of $GL(\mathcal{H})$ given by

$$\mathcal{U}(\mathcal{H}) := \{A \in GL(\mathcal{H}) | (A(x), A(y)) = (x, y) \text{ for all } x, y \in \mathcal{H}\}$$

so $A \in \mathcal{U}(\mathcal{H})$ if and only if $A^*A = AA^* = id_{\mathcal{H}}$, i.e., $A^{-1} = A^*$. One has the following:

Theorem. $\mathcal{U}(\mathcal{H})$ is a closed, regular, real Banach submanifold of $GL(\mathcal{H})$.

Corollary. $\mathcal{U}(\mathcal{H})$ is a closed, real Banach Lie subgroup of $GL(\mathcal{H})$. In particular, $\mathcal{U}(\mathcal{H})$ is a real Banach Lie group.

Proof of the Theorem. Let f: $GL(\mathcal{H}) \to \mathcal{L}^s(\mathcal{H})$ be given by $f(A) := A^*A$; then $f^{-1}(Id) = \mathcal{U}(\mathcal{H})$. $GL(\mathcal{H})$ is open in $\mathcal{L}(\mathcal{H})$ and f is the restriction to

$GL(\mathcal{H})$ of the smooth function $\tilde{f}$: $\mathcal{L}(\mathcal{H}) \to \mathcal{L}^s(\mathcal{H}), \tilde{f}(A) := f(A)$, so f is smooth. Its derivative at A in the direction $T \in \mathcal{L}(\mathcal{H})$ is

$$Df(A)(T) = \lim_{t \to 0}(f(A + tT) - f(A))/t = A^*T + T^*A$$

so $\ker(Df(A)) = \{T \in \mathcal{L}(\mathcal{H}) | A^*T = -T^*A\}$ and in particular $\ker(Df(Id)) = \{T \in \mathcal{L}(\mathcal{H}) | T^* = -T\} = \mathcal{L}^a(\mathcal{H})$, which splits in $\mathcal{L}(\mathcal{H})$; also $Df(Id)$: $\mathcal{L}(\mathcal{H}) \to \mathcal{L}^s(\mathcal{H})$ is onto since for $B \in \mathcal{L}^s(\mathcal{H})$, $Df(Id)(\frac{1}{2}B) = \frac{1}{2}B + \frac{1}{2}B^* = B$. Then $Id \in \mathcal{L}^s(\mathcal{H})$ is a regular value of f, and by the implicit function theorem, $\mathcal{U}(\mathcal{H})$ is a closed, regular, real Banach submanifold of $GL(\mathcal{H})$, with $T_A\mathcal{U}(\mathcal{H}) = \ker(Df(A))$ and in particular $T_{Id}\mathcal{U}(\mathcal{H}) = \mathcal{L}^a(\mathcal{H})$, which are real Banach spaces. QED

The restriction of the *action* $GL(H) \times S(\mathcal{H})$, $(T, x) \mapsto T(x)$, to $\mathcal{U}(\mathcal{H})$ is the smooth action α: $\mathcal{U}(\mathcal{H}) \times S(\mathcal{H}) \to S(\mathcal{H})$, $\alpha(T, x) = T(x)$. For a fixed $x_0 \in S(\mathcal{H})$, α induces the smooth function α_{x_0}: $\mathcal{U}(\mathcal{H}) \to S(\mathcal{H})$, $T \mapsto \alpha_{x_0}(T) = T(x_0)$ [$\alpha_{x_0}(\mathcal{U}(\mathcal{H}))$ is the orbit of x_0 under $\mathcal{U}(\mathcal{H})$] and one has the *isotropy* subgroup at x_0, $\mathcal{U}(\mathcal{H})_{x_0} = \{T \in \mathcal{U}(\mathcal{H}) | T(x_0) = x_0\} = \alpha_{x_0}^{-1}(\{x_0\})$; clearly, $\mathcal{U}(\mathcal{H})_{x_0}$ is a closed subgroup of $\mathcal{U}(\mathcal{H})$. In the following, and without loss of generality, we shall take $x_0 = e_1$.

It is easy to verify that α_{e_1} is surjective or, in other words, that the action α is transitive: in fact, let $h = ze_1 \in S(\mathcal{H})$ (this implies $|z| = 1$); if $T \in \mathcal{L}(\mathcal{H})$ is given by $T(v) = zv$, one has $\|T(v)\| = \|zv| = \|v\|$ and then $T \in \mathcal{U}(\mathcal{H})$, i.e., $T \in \alpha_{e_1}^{-1}(\{h\})$; now consider again $h \in S(\mathcal{H})$, but h and e_1 linearly independent; then e_1 and h span a 2-dimensional complex closed subspace V of $\mathcal{H}$, and applying twice the Gram–Schmidt orthonormalization process, we obtain the pair of orthonormal bases of V, $\{e_1, \tilde{h}\}$ and $\{\tilde{e}_1, h\}$, with

$$\tilde{h} = (h - (h, e_1)e_1)/\|h - (h, e_1)e_1\|, \qquad \tilde{e}_1 = (e_1 - (e_1, h)h)/\|e_1 - (e_1, h)h\|$$

Then the operator A: $V \to V$ defined as the complex linear extension of $A(e_1) = h$ and $A(\tilde{h}) = \tilde{e}_1$ is unitary and one can verify that $A \oplus id_{V^\perp}$: $V \oplus V^\perp \to V \oplus V^\perp$ is linear, bounded, and unitary on $\mathcal{H} = V \oplus V^\perp$, so $A \oplus id_{V^\perp} \in \alpha_{e_1}^{-1}(\{h\})$. Then $S(\mathcal{H})$ is a homogeneous space and one has the commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\quad \alpha_{e_1} \quad} & S(\mathcal{H}) \\ & {\scriptstyle p} \searrow \qquad \nearrow {\scriptstyle \bar{\alpha}_{e_1}} & \\ & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} & \end{array}$$

where p is the projection $p(T) = [T] = T\mathcal{U}(\mathcal{H})_{e_1}$ and $\bar{\alpha}_{e_1}$ is the bijection $\bar{\alpha}_{e_1}([T]) = T'(e_1)$ with $T' \in [T]$. In order to be able to apply Theorem A (see Appendix), we have to verify that $\mathcal{U}(\mathcal{H})_{e_1}$ is a regular Banach Lie subgroup of $\mathcal{U}(\mathcal{H})$; from the implicit function theorem, that is a consequence of e_1 being a regular value of α_{e_1}. It is sufficient to show that

$$D\alpha_{e_1}(Id): T_{Id}\mathcal{U}(\mathcal{H}) = \mathcal{L}^a(\mathcal{H}) \to T_{e_1}S(\mathcal{H})$$
$$= \{v \in \mathcal{H}, v = (z_1, z_2, \ldots), z_1 \text{ imaginary}\}$$
$$T \mapsto D\alpha_{e_1}(Id)(T) = \alpha_{e_1}(T) = T(e_1)$$

is *surjective* and has a *split kernel* $\ker(D\alpha_{e_1}(Id)) = \{T \in \mathcal{L}^a(\mathcal{H}) | T(e_1) = 0\}$, which is closed in $\mathcal{L}^a(\mathcal{H})$.

Let $v = (z_1, z_2, \ldots)$ be a fixed vector in $\mathcal{H}$; then, it is easy to show that the maps l_v: $\mathcal{H} \to \mathcal{H}$, f_v: $\mathcal{H} \to \mathbb{C}$, and L_v: $\mathcal{H} \to \mathbb{C}$, respectively, given by $l_v(h) = h_1 v$, $f_v(h) = \bar{z}_1 h_1$, and $L_v(h) = (h, v)$ are complex linear and bounded (continuous); then $T_v = l_v + \iota \circ (f_v - L_v)$: $\mathcal{H} \to \mathcal{H}$ with ι: $\mathbb{C} \to \mathcal{H}$, $\iota(z) := ze_1$ is linear and continuous and gives $T_v(h) = h_1 v + (h_1\bar{z}_1 - (h, v))e_1$; in particular, $T_v(e_1) = v$ and $T_v(e_i) = -\bar{z}_i e_1$ for $i > 1$, so $(T_v(e_1), e_r) = (v, e_r) = z_r$, $r \leq 1$, and $(T_v(e_i), e_r) = -\bar{z}_i(e_1, e_r)$ $(= -\bar{z}_i$ for $r = 1$ and $= 0$ for $r > 1)$ for $i > 1$. Then it is easy to verify that if $v \in T_{e_1}S(\mathcal{H})$, the adjoint operator T_v^* defined by $(T_v^*(x), y) = (x, T_v(y))$ satisfies $((T_v + T_v^*)(e_i), e_r) = 0$ for all $i, r \geq 1$, and since $\{e_i, i = 1, 2, \ldots\}$ is a complete orthonormal set, $T_v^* = -T_v$, i.e., $T_v \in \mathcal{L}^a(\mathcal{H})$ and so $D\alpha_{e_1}(Id)$ is surjective.

Define the function φ: $T_{e_1}S(\mathcal{H}) \to T_{Id}\mathcal{U}(\mathcal{H})$, $v \mapsto \varphi(v) := T_v$; then φ is a right inverse of $D\alpha_{e_1}(Id)$ since

$$D\alpha_{e_1}(Id) \circ \varphi = id_{T_{e_1}S(\mathcal{H})}$$

and is real linear and injective, so φ is an excision of the short exact sequence

$$0 \to \ker(D\alpha_{e_1}(Id)) \overset{\iota}{\to} T_{Id}\mathcal{U}(\mathcal{H}) \xrightarrow{D\alpha_{e_1}(Id)} T_{e_1}S(\mathcal{H}) \to 0$$

which therefore splits, i.e., one has the isomorphism of topological vector spaces

$$\psi: \ker(D\alpha_{e_1}(Id)) \oplus T_{e_1}S(\mathcal{H}) \to T_{Id}\mathcal{U}(\mathcal{H}), \psi(T \oplus v)$$
$$= \iota(T) + \varphi(v) = T + T_v$$

Notice, however, that $Im(\varphi)$ is not the topological complement of $\ker(D\alpha_{e_1}(Id))$ in $\mathcal{L}^a(\mathcal{H})$, since $Im(\varphi) \cap \ker(D\alpha_{e_1}(Id)) = \{0\} \neq \phi$. Consider a sequence T_{v_n} in $Im(\varphi)$ which converges to T in $\mathcal{L}^a(\mathcal{H})$ as $n \to \infty$; to prove that $Im(\varphi)$ is closed in $\mathcal{L}^a(\mathcal{H})$ amounts to showing that $T = T_v$ for some $v \in T_{e_1}S(\mathcal{H})$. Since $T_{v_n} \to T$, then T_{v_n} is a Cauchy sequence and therefore for any fixed $\lambda \in \mathcal{H}$,

$$\|T_{v_n}(h) - T_{v_m}(h)\| = \|(T_{v_n} - T_{v_m})(h)\| \leq \|T_{v_n} - T_{v_m}\| \cdot \|h\| < \delta$$

for $n, m \geq N_0$; then $T_{v_n}(h)$ is a Cauchy sequence in $\mathcal{H}$ and by completeness $T_{v_n}(h) \to \bar{h}$; in particular for $h = e_1$, $T_{v_n}(e_1) = v_n \to \bar{h} \equiv v \in T_{e_1}S(\mathcal{H})$. Using the result that if $T_{v_n} \to T$ then $T_{v_n}(h) \to T(h)$ for any $h \in \mathcal{H}$, if $v = (z_1, z_2, \ldots)$ and $v_n = (z_{n_1}, z_{n_2}, \ldots)$, we have

$$T(e_j) = \lim_n T_{v_n}(e_j) = -\lim_n \bar{z}_{n_j} e_1 = -\bar{z}_j e_1 \qquad \text{for} \quad j \geq 2$$

$$T(e_1) = \lim_n T_{v_n}(e_1) = \lim_n v_n = v, \qquad T_v(e_j) = -\bar{z}_j e_1 \qquad \text{for} \quad j \geq 2$$

and $T_v(e_1) = v$, so $T = T_v \in Im(\varphi)$. From the implicit function theorem one also has $T_{Id}\mathcal{U}(\mathcal{H})_{e_1} = Lie(\mathcal{U}(\mathcal{H})_{e_1}) = \ker(D\alpha_{e_1}(Id))$.

We now apply Theorem A (see Appendix) with the identifications $G = \mathcal{U}(\mathcal{H})$, $H = \mathcal{U}(\mathcal{H})_{e_1}$, $h = \ker(D\alpha_{e_1}(Id))$, and $l = Im(\varphi)$: There is a unique differentiable structure on $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$ such that the projection p: $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$ is a submersion. With this structure, the canonical action $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$ is smooth and

$$Dp(Id)|_{Im(\varphi)}: \quad Im(\varphi) \to T_{[Id]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1})$$

is an isomorphism of topological vector spaces.

To prove that $\bar{\alpha}_{e_1}$ is a diffeomorphism of Banach manifolds we use some properties of submersions (Abraham *et al.*, 1988; Aguilar, 1996). Let M, N, and Q be Banach manifolds, f: $M \to N$ a surjective submersion, and g: $M \to Q$ and $\bar{g}$: $N \to Q$ functions such that $\bar{g} \circ f = g$. Then g is smooth if and only if $\bar{g}$ is smooth. (This is a consequence of the fact that a surjective submersion has smooth local sections.) Then $\bar{\alpha}_{e_1}$ is smooth: in fact, identify $M = \mathcal{U}(\mathcal{H})$, $N = \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$, $Q = S(\mathcal{H})$, $f = p$, $g = \alpha_{e_1}$, and $\bar{g} = \bar{\alpha}_{e_1}$.

The proof that $\bar{\alpha}_{e_1}^{-1}$ is smooth is based on the *inverse function theorem*, which is valid for Banach manifolds and says that if the differential of a smooth bijective function f: $M \to N$ is a topological vector space isomorphism at all points of its domain, then f is a diffeomorphism, i.e., f^{-1} is smooth. The derivative of $\bar{\alpha}_{e_1} \circ p = \alpha_{e_1}$ at $Id \in \mathcal{U}(\mathcal{H})$ is represented by the commutative diagram

$$\begin{array}{ccc} T_{Id}\mathcal{U}(\mathcal{H}) & \xrightarrow{\quad D\alpha_{e_1}(Id) \quad} & T_{e_1}S(\mathcal{H}) \\ & {\scriptstyle Dp(Id)} \searrow \qquad \nearrow {\scriptstyle D\bar{\alpha}_{e_1}([Id])} & \\ & T_{[Id]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}) & \end{array}$$

When restricted to $Im(\varphi)$, $Dp(Id)$ becomes a topological vector space isomorphism by Theorem A, and the analogous thing happens to $D\alpha_{e_1}(Id)$ since it is a left inverse of φ, which is injective, so

$$D\overline{\alpha}_{e_1}([Id]) = D\alpha_{e_1}(Id)|_{Im(\varphi)} \circ Dp(Id)|^{-1}_{Im(\varphi)}$$

is a topological vector space isomorphism. Fix now a unitary operator A and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\alpha_{e_1}} & S(\mathcal{H}) \\ {\scriptstyle L_A}\downarrow & & \downarrow{\scriptstyle \overline{A}} \\ \mathcal{U}(\mathcal{H}) & \xrightarrow[\alpha_{e_1}]{} & S(\mathcal{H}) \end{array}$$

where the left translation L_A and $\overline{A}$ are diffeomorphisms given by $L_A(T) = A \circ T$ and $\overline{A}(v) = A(v)$, respectively. Passing to the quotients, we obtain

$$\begin{array}{ccccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{p} & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} & \xrightarrow{\overline{\alpha}_{e_1}} & S(\mathcal{H}) \\ {\scriptstyle L_A}\downarrow & & {\scriptstyle \overline{L}_A}\downarrow & & \downarrow{\scriptstyle \overline{A}} \\ \mathcal{U}(\mathcal{H}) & \xrightarrow[p]{} & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} & \xrightarrow[\overline{\alpha}_{e_1}]{} & S(\mathcal{H}) \end{array}$$

where $\overline{L}_A([T]) = [A \circ T]$. Since $\overline{L}_A \circ p = p \circ L_A$ is smooth and p is a submersion, then $\overline{L}_A$ is smooth with smooth inverse $\overline{L}_A^{-1} = \overline{L}_{A^*}$; then we can take differentials and obtain

$$\begin{array}{ccccc} T_{Id}\mathcal{U}(\mathcal{H}) & \xrightarrow{Dp(Id)} & T_{[Id]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}) & \xrightarrow{D\overline{\alpha}_{e_1}([Id])} & T_{e_1}S(\mathcal{H}) \\ {\scriptstyle DL_A(Id)}\downarrow & & {\scriptstyle D\overline{L}_A([Id])}\downarrow & & \downarrow{\scriptstyle D\overline{A}([e_1])} \\ T_A\mathcal{U}(\mathcal{H}) & \xrightarrow[Dp(A)]{} & T_{[A]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}) & \xrightarrow[D\overline{\alpha}_{e_1}([A])]{} & T_{A(e_1)}S(\mathcal{H}) \end{array}$$

The right part of the diagram shows that $D\overline{\alpha}_{e_1}([A])$ is a composition of topological vector space isomorphisms and therefore is a topological vector space isomorphism. Then $\overline{\alpha}_{e_1}$ is a diffeomorphism and so $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$ and $S(\mathcal{H})$ are diffeomorphic as Banach manifolds. *The differentiable structure on $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$ determined by Theorem A is nothing but the diffeomorphic translation of the differentiable structure on $S(\mathcal{H})$ as a submanifold of $\mathcal{H}$ by $\overline{\alpha}_{e_1}^{-1}$.*

4.4. The Canonical Decomposition of $T_{A(e_1)}S(\mathcal{H})$

The canonical decomposition was defined in Section 4.2 at $e_1 \in S(\mathcal{H})$ as the subspace H_{e_1} of $T_{e_1}S(\mathcal{H})$. Through the isomorphism $D\overline{\alpha}_{e_1}([Id])$ we push it forward to the class of the identity in the homogeneous manifold $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$:

$$H_{[Id]} := D\bar{\alpha}_{e_1}([Id])^{-1}(H_{e_1}) = Dp(Id)|_{Im(\varphi)} \circ D\alpha_{e_1}(Id)|^{-1}_{Im(\varphi)}(H_{e_1})$$

$$= Dp(Id)|_{Im(\varphi)} (\{T_v \in \mathcal{L}^a(\mathcal{H})|v = (0, z_2, \ldots) \in \mathcal{H}\}) := Dp(Id)|_{Im(\varphi)}(H_{Id})$$

Thus there is a canonical topological vector space isomorphism between the horizontal space at $[Id]$ and the Banach subspace H_{Id} of the skew-Hermitian operators [i.e., of $T_{Id}\mathcal{U}(\mathcal{H})$] consisting of the operators T_v with zero first component of v. For the vertical space at the same point we have

$$V_{[Id]} = Dp(Id)|_{Im(\varphi)}(\{T_v \in \mathcal{L}^a(\mathcal{H})|v = (i\lambda, 0, \ldots), \lambda \in \mathbb{R}\})$$

with $T_{[Id]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}) = H_{[Id]} \oplus_{\mathbb{R}} V_{[Id]}$, and defining $V_{Id} = \{T_v \in \mathcal{L}^a(\mathcal{H})|v = (i\lambda, 0, \ldots), \lambda \in \mathbb{R}\}$, we arrive at $T_{Id}\mathcal{U}(\mathcal{H}) = H_{Id} \oplus_{\mathbb{R}} V_{Id}$.

To obtain the decomposition at all points we consider the left part of the last diagram. There $DL_A(Id)$ is a topological vector space isomorphism, and since $\mathcal{U}(\mathcal{H}) \subset GL(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$, then $DL_A(Id) = L_A$ and so $DL_A(Id)(T) = L_A(T) = A \circ T$. Then the horizontal and the vertical spaces at $A \in \mathcal{U}(\mathcal{H})$ are respectively given by $H_A = \{A \circ T_v|v = (0, z_2, \ldots) \in \mathcal{H}\}$ and $V_A = \{A \circ T_v|v = (i\lambda, 0, \ldots) \in \mathcal{H}\}$ with $H_A \oplus V_A = T_A\mathcal{U}(\mathcal{H})$.

The remarkable point is that *we can "see" the universal connection* (after Section 4.6) *on the projective Hilbert bundle at the tangent spaces of the unitary group of the Hilbert space.* To "see" the connection at the tangent space of the homogeneous manifold $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$, we use the fact that the restriction $Dp(A)|$ of $Dp(A)$ to the image of $Im(\varphi)$ in $T_{Id}\mathcal{U}(\mathcal{H})$ by $DL_A(Id)$ [i.e., to $A \circ Im(\varphi)$] can be shown to be a topological vector space isomorphism. Then we have the horizontal and vertical spaces $H_{[A]} = Dp(A)|(H_A)$ and $V_{[A]} = Dp(A)|(V_A)$, satisfying $H_{[A]} \oplus_{\mathbb{R}} V_{[A]} = T_{[A]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1})$. We can return to the Hilbert sphere and have the connection and vertical space at a point $A_{(e_1)}$ given by $H_{A(e_1)} = D\bar{\alpha}_{e_1}([A])(H_{[A]})$ and $V_{A(e_1)} = D\bar{\alpha}_{e_1}([A])(V_{[A]})$, with $H_{A(e_1)} \oplus_{\mathbb{R}} V_{A(e_1)} = T_{A(e_1)}S(\mathcal{H})$.

A final remark concerns the fact that the decomposition is well defined. Let A and B be two different elements in $\mathcal{U}(\mathcal{H})$ but belonging to the same class in $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1}$, i.e., $[A] = [B]$. Obviously H_A and H_B are different, and the same holds for V_A and V_B. However, they are two distinct representations on the tangent bundle of $\mathcal{U}(\mathcal{H})$ of the *same* decomposition, since, as can be easily shown, $Dp(A) \circ DL_A(Id) = Dp(B) \circ DL_B(Id)$ and then $Dp(A)|(H_A) = Dp(B)|(H_B)$; the analogous result holds for the vertical spaces.

4.5. $\mathcal{P}(\mathcal{H})$ as a Homogeneous Space of Banach Lie Groups

There are two topologically equivalent definitions of the projective Hilbert space. The first definition consists of the complex one-dimensional subspaces of $\mathcal{H}$ (lines through the origin in $\mathcal{H}$) and is given by $\overline{\mathcal{P}}(\mathcal{H}) := \mathcal{H}^*/\mathbb{C}^* = \{[v]' = v\mathbb{C}^*, v \in \mathcal{H}^*\}$, where $\mathcal{H}^* = \mathcal{H} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$, while the second definition (already given at the beginning of this section) consists of the orbit space of the action $S(\mathcal{H}) \times S^1 \to S(\mathcal{H})$, $(v, w) \mapsto vw$, $\mathcal{P}(\mathcal{H}) = S(\mathcal{H})/S^1 = \{[v] = vS^1, v \in S(\mathcal{H})\}$. Using the projections p: $S(\mathcal{H}) \to \overline{\mathcal{P}}(\mathcal{H})$, $p(v) = [v]'$, and q: $S(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$, $q(v) = [v]$, we have that $\overline{\mathcal{P}}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ are given the quotient topology, so p and q are identifications and q is open and closed. [A continuous function f: $X \to Y$ is an *identification* if, whenever $f^{-1}(V)$ is open in X, then V is open in Y. An identification has the following property: given β: $Y \to Z$ such that $\beta \circ f = \alpha$: $X \to Z$ is continuous, then β is continuous.] Then it is easy to verify that ψ: $\overline{\mathcal{P}}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$, $\psi(v\mathbb{C}^*) = v/\|v\|S^1$ is a homeomorphism: in fact ψ is a bijection which satisfies $\psi \circ p = q$, and if V is open in $\mathcal{P}(\mathcal{H})$, then $q^{-1}(V) = p^{-1}(\psi^{-1}(V))$ is open in $S(\mathcal{H})$ and so $\psi^{-1}(V)$ is open in $\overline{\mathcal{P}}(\mathcal{H})$, i.e., ψ is continuous; analogously one proves that ψ^{-1} is continuous. Then p is open and closed. These results are summarized in the following commuting diagram:

$$\begin{array}{ccc} & S(\mathcal{H}) & \\ {}^{p}\swarrow & & \searrow^{q} \\ \overline{\mathcal{P}}(\mathcal{H}) & \xrightarrow[\psi]{} & \mathcal{P}(\mathcal{H}) \end{array}$$

The action α: $\mathcal{U}(\mathcal{H}) \times S(\mathcal{H}) \to S(\mathcal{H})$ passes to the quotient and one has the commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) \times S(\mathcal{H}) & \xrightarrow{\alpha} & S(\mathcal{H}) \\ {\scriptstyle id \times q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{U}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) & \xrightarrow[\beta]{} & \mathcal{P}(\mathcal{H}) \end{array}$$

with $\beta(A, [v]) = [Av]$. Here β is continuous, since $id \times q$ is an identification. The isotropy group of $[e_1] \in \mathcal{P}(\mathcal{H})$ is the subgroup of $\mathcal{U}(\mathcal{H})$ given by

$$\mathcal{U}(\mathcal{H})_{[e_1]} = \{A \in \mathcal{U}(\mathcal{H}) | A[e_1] = [e_1]\} = \{A \in \mathcal{U}(\mathcal{H}) | Ae_1 = we_1, w \in S^1\}$$

Clearly $\mathcal{U}(\mathcal{H})_{e_1} \subset \mathcal{U}(\mathcal{H})_{[e_1]}$. As was the case for $S(\mathcal{H})$, one has the commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\beta_{[e_1]}} & \mathcal{P}(\mathcal{H}) \\ {}_{\pi}\searrow & & \nearrow_{\overline{\beta}_{[e_1]}} \\ & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]} & \end{array}$$

where π is the projection $\pi(T) = [T] = T\mathcal{U}(\mathcal{H})_{[e_1]}$, $\beta_{[e_1]}(T) = \beta(T, [e_1])$, and $\bar{\beta}_{[e_1]}$ is the continuous bijection $\bar{\beta}_{[e_1]}([T]) = \beta_{[e_1]}(T')$ with $T' \in [T]$. It is easy to verify that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\alpha_{e_1}} & S(\mathcal{H}) \\ & {\scriptstyle \beta_{[e_1]}} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathcal{P}(\mathcal{H}) & \end{array}$$

Since α_{e_1} and q are open, then $\beta_{[e_1]}$ is open; so $\bar{\beta}_{[e_1]}$ is open: in fact, if $V = \pi(\pi^{-1}(V))$ is open in $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H}_{[e_1]})$, then $\beta_{[e_1]}(V) = \beta_{[e_1]}(\pi^{-1}(V))$ is open in $\mathcal{P}(\mathcal{H})$. So $\bar{\beta}_{[e_1]}$ is a *homeomorphism.*

Define the function γ: $\mathcal{U}(\mathcal{H})_{e_1} \times S^1 \to \mathcal{U}(\mathcal{H})_{[e_1]}$, $(T, w) \mapsto \gamma(T, w)$: $\mathcal{H} \to \mathcal{H}$, $\gamma(T, w)(e_1) = we_1$, and $\gamma(T, w)(e_i) = Te_i$, $i > 1$ [it is easy to verify that $(we_1, Te_2, \ldots)$ is orthonormal, then $\gamma(T, w)$ is linear and continuous]; γ is a *bijection* with an inverse given by γ^{-1}: $\mathcal{U}(\mathcal{H})_{[e_1]} \to \mathcal{U}(\mathcal{H})_{[e_1]} \times S^1$, $\gamma^{-1}(A) = (T_A, w)$, where w is determined by $Ae_1 = we_1$ and $T_Ae_1 = e_1$ and $T_Ae_i = Ae_i$ [again $(e_i, Ae_2, \ldots)$ is orthonormal]. Also, γ is a group homomorphism (isomorphism). We have the following result.

Proposition. The function (group homomorphism) ι: $S^1 \to \mathcal{U}(\mathcal{H})$, $\iota(w) := \gamma(Id, w)$: $\mathcal{H} \to \mathcal{H}$, $\iota(w)(e_1) = we_1$, $\iota(w)(e_i) = e_i$, $i > 1$, is an embedding from $S^1 \to \mathcal{U}(\mathcal{H})$, i.e., S^1 is a regular Banach Lie subgroup of $\mathcal{U}(\mathcal{H})$.

Proof. Clearly ι is injective. Let f: $\mathbb{C} \to \mathcal{L}(\mathcal{H})$, $f(z) = \lambda_z \oplus id$: $\langle e_1 \rangle \oplus \langle e_1 \rangle^\perp = \mathcal{H} \to \mathcal{H}$ be given by $\lambda_z(e_1) = ze_1$; f is an extension of ι since $f|_{S^1} = \iota$ and from the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathcal{L}(\mathcal{H}) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ S^1 & \xrightarrow[\iota]{} & \mathcal{U}(\mathcal{H}) \end{array}$$

where the inclusions i and j are regular embeddings, ι is *smooth* since $i \circ \iota = f \circ j$ is smooth. The derivative of f, $Df(z)$: $\mathbb{C} \to \mathcal{L}(\mathcal{H})$, in the direction of $v \in \mathbb{C}$ at $h = z'e_1 \oplus y \in \mathcal{H}$ is given by

$$Df(z)(v)(h) = \lim_{t\to 0} \frac{1}{t} (f(z + tv) - f(z))(z'e_1 \oplus y)$$

$$= \lim_{t\to 0} \frac{1}{t} ((z + tv)z'e_1 \oplus y - (zz'e_1 \oplus y)) = vz'e_1$$

independent of z; if $Df(z)(v) = 0$, then $Df(z)(v)(e_1) = ve_1 = 0$, i.e., $v = 0$, and so $Df(z)$ is injective at each z; taking differentials in the last diagram, one obtains the derivative of ι, $D\iota(w)$: $T_wS^1 \to T_{\iota(w)}\mathcal{U}(\mathcal{H})$, which is injective at each $w \in S^1$. To prove that $D\iota(w)(T_wS^1)$ is a split subspace of $T_{\iota(w)}\mathcal{U}(\mathcal{H})$ at each $w \in S^1$, it is enough to prove that at $w = 1$

$$D\iota(1)(T_1S^1) = \{D\iota(1)(i\lambda),\ \lambda \in \mathbb{R}\} \subset T_{Id}\mathcal{U}(\mathcal{H}) = \mathcal{L}^a(\mathcal{H})$$

with

$$D\iota(1)(i\lambda) \equiv \rho_\lambda\colon \mathcal{H} \to \mathcal{H}, \qquad \rho_\lambda(z'e_1 \oplus y) = i\lambda z'e_1$$

[in particular, $\rho_\lambda(e_1) = i\lambda e_1$ and $\rho_\lambda(e_k) = 0$, $k > 1$]; notice that $\rho_\lambda = \lambda\rho_1$ and so $D\iota(1)(T_1S^1) = \langle\rho_1\rangle$, which is closed in $\mathcal{L}^a(\mathcal{H})$; according to Theorem A in the Appendix, its (closed) complement (when it exists) with respect to $\mathcal{L}^a(\mathcal{H})$ is the kernel of a linear continuous operator P: $\mathcal{L}^a(\mathcal{H}) \to \mathcal{L}^a(\mathcal{H})$ satisfying $P \circ P = P$ and such that the set of its fixed points is $D\iota(1)(T_1S^1)$. Let $P(A) := \varphi(A)\rho_1$, where $\varphi(A) = 1/i(Ae_1, e_1) \in \mathbb{R}$ [since $A \in \mathcal{L}^a(\mathcal{H})$, (Av, v) is pure imaginary for any $v \in \mathcal{H}$]; in particular, $\varphi(\rho_1) = 1/i(\rho_1e_1, e_1) = 1/i(ie_1, e_1) = 1$. One has: φ is linear and continuous (it is the composition of the evaluation and the inner product);

$$P^2(A) = P(\varphi(A)\rho_1) = \varphi(A)\varphi(\rho_1)\rho_1 = \varphi(A)\rho_1 = P(A)$$

Let $\rho_\lambda \in \langle\rho_1\rangle$; then $P(\rho_\lambda) = P(\lambda\rho_1) = \lambda P(\rho_1) = \lambda\rho_1$, i.e., ρ_λ is a fixed vector of P; if $A \in \mathcal{L}^a(\mathcal{H})$ is a fixed vector of P, then $P(A) = \varphi(A)\rho_1 = A$, i.e., $A \in \langle\rho_1\rangle$, so {fixed vectors of P} $= D\iota(1)(T_1S^1)$. Then $\langle\rho_1\rangle$ is a split subspace of $\mathcal{L}^a(\mathcal{H})$, i.e., $\mathcal{L}^a(\mathcal{H}) = D\iota(1)(T_1S^1) \oplus \ker(P)$ with $\ker(P) = \{A \in \mathcal{L}^a(\mathcal{H}) | (Ae_1, e_1) = 0\}$. Finally, ι is closed since S^1 is compact and $\mathcal{U}(\mathcal{H})$ is Hausdorff. Then ι is an embedding. QED

Define the function $i\cdot\iota := \mu \circ (i \times \iota)$, where μ is the composition in $\mathcal{U}(\mathcal{H})$:

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H})_{e_1} \times S^1 & \xrightarrow{i\cdot\iota} & \mathcal{U}(\mathcal{H}) \\ & {\scriptstyle i\times\iota}\searrow \quad \nearrow{\scriptstyle \mu} & \\ & \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) & \end{array}$$

It is easy to verify that $i\cdot\iota(T, w) = \gamma(T, w)$, so $i\cdot\iota$ is a bijection onto its image, $i\cdot\iota(\mathcal{U}(\mathcal{H})_{e_1} \times S^1) = \mathcal{U}(\mathcal{H})_{[e_1]}$. We have the following result.

Proposition. $i\cdot\iota$ is an embedding from $\mathcal{U}(\mathcal{H})_{e_1} \times S^1$ to $\mathcal{U}(\mathcal{H})$, i.e., $\mathcal{U}(\mathcal{H})_{e_1} \times S^1$ (or $\mathcal{U}(\mathcal{H})_{[e_1]}$) is a regular submanifold of $\mathcal{U}(\mathcal{H})$.

Proof. $\mathcal{U}(\mathcal{H})_{e_1}$ and $\iota(S^1)$ are regular Banach Lie subgroups of $\mathcal{U}(\mathcal{H})$, and they commute with each other, i.e., $\iota(w) \circ T = T \circ \iota(w)$ for all $w \in S^1$ and $T \in \mathcal{U}(\mathcal{H})_{e_1}$ [in fact, $T \circ \iota(w)(e_1) = T(we_1) = wTe_1 = we_1$, $\iota(w) \circ T(e_1) = \iota(w)(e_1) = we_1$, and, for $k > 1$, $T \circ \iota(w)(e_k) = Te_k$ and $\iota(w) \circ T(e_k) = Te_k$, since $(e_1, Te_k) = (Te_1, Te_k) = (e_1, e_k) = 0$ and then $Te_k = \sum_{j=2}^{\infty} \alpha_{kj}e_j$, which is invariant under $\iota(w)$]; so $i \cdot \iota$ is a smooth monomorphism since

$$\begin{aligned} i \cdot \iota(T_1 \circ T_2, w_1w_2) & \\ &= \mu \circ (i \times \iota)(T_1 \circ T_2, w_1w_2) \\ &= i(T_1 \circ T_2) \circ \iota(w_1w_2) = (T_1 \circ T_2) \circ (\iota(w_1) \circ \iota(w_2)) \\ &= (T_1 \circ \iota(w_1)) \circ (T_2 \circ \iota(w_2)) = i \cdot \iota(T_1, w_1) \circ i \cdot \iota(T_2, w_2) \end{aligned}$$

and if $i \cdot \iota(T, w) = Id$, then $T \circ \iota(w)(e_1) = wTe_1 = we_1 = e_1$ and for $k > 1$, $T \circ \iota(w)(e_k) = Te_k = e_k$, so $(T, w) = (Id, 1)$. We compute now the differential of $i \cdot \iota$: $D(\mu \circ (i \times \iota)) = D\mu \circ (Di \times D\iota)$; again it is enough to study that function at the identity. We use the result that the differential of the composition in a Banach Lie group G, $\mu: G \times G \to G$ is given by $D\mu(g_1, g_2)(v_1, v_2) = g_1 \circ v_2 + v_1 \circ g_2$; in particular, $D\mu(Id, Id)(v_1v_2) = v_1 + v_2$. Consider the diagram

$$\begin{array}{ccc} T_{Id}\mathcal{U}(\mathcal{H})_{e_1} \times T_1S^1 & \xrightarrow{Di(Id) \times D\iota(1)} & T_{Id}\mathcal{U}(\mathcal{H}) \times T_{Id}\mathcal{U}(\mathcal{H}) \\ & {\scriptstyle D(i\cdot\iota)(Id,\ 1)} \searrow & \big\downarrow {\scriptstyle D\mu(Id,\ 1)} \\ & & T_{ID}\mathcal{U}(\mathcal{H}) \end{array}$$

and recall that $T_{Id}\mathcal{U}(\mathcal{H}) = \mathcal{L}^a(\mathcal{H})$ and $T_{Id}\mathcal{U}(\mathcal{H})_{e_1} = \{T \in \mathcal{L}^a(\mathcal{H}) | Te_1 = 0\}$. Then

$$\begin{aligned} D(i \cdot \iota)(Id, 1)(T, i\lambda) & \\ &= D\mu(Id, 1)(Di(Id)(T), D\iota(1)(i\lambda)) = D\mu(Id, 1)(T, \rho_\lambda) \\ &= T + \rho_\lambda: \mathcal{H} \to \mathcal{H}, \qquad z'e_1 \oplus y \mapsto T(z'e_1 \oplus y) + \rho_\lambda(z'e_1 \oplus y) \\ &= Ty + i\lambda z'e_1 \end{aligned}$$

In particular, $(T + \rho_\lambda)(e_1) = i\lambda e_1$ and $(T + \rho_\lambda)(e_k) = Te_k$ for $k > 1$; if $T + \rho_\lambda = 0$, then $\lambda = 0$ and $T = 0$, i.e., $ker(D(i \cdot \iota)(Id, 1)) = \{(0, 0)\}$ and therefore $D(i \cdot \iota)(Id, 1)$ is *injective*. To prove that its image splits, we use Proposition B in the Appendix, and identify $V = \mathcal{L}^a(\mathcal{H})$, $H = T_{Id}\mathcal{U}(\mathcal{H})_{e_1}$, $H' = \{T_v \in \mathcal{L}^a(\mathcal{H}) | v = (z_1, z_2, \ldots), z_1 \text{ imaginary}\}$, $K = D\iota(1)(T_1S^1) = \{\rho_\lambda, \lambda \in \mathbb{R}\}$, and $K' = \{A \in \mathcal{L}^a(\mathcal{H}) | (Ae_1, e_1) = 0\}$, which obey the required conditions, in particular $H \subset K'$ (trivial), for $K \subset H'$: $\rho_\lambda = T_{i\lambda e_1}$ [in fact, from $T_ve_1 = v$

and $T_v e_k = -\bar{z}_k e_k$ for $k > 1$, where $v = (z_1, z_2, \ldots)$ we have $T_{i\lambda e_1} e_1 = i\lambda e_1$ and $T_{i\lambda e_1} e_r = 0$], and if $\rho_\lambda \in H$, then $\rho_\lambda(e_1) = 0$ and so $\lambda = 0$, i.e., $\rho_\lambda = 0$. Then

$$\mathcal{L}^a(\mathcal{H}) = (T_{Id}\mathcal{U}(\mathcal{H})_{e_1} \oplus D\iota(1)(T_1 S^1)) \oplus H' \cap K'$$

where the complement $H' \cap K'$ is given by

$$\{T_v \in \mathcal{L}^a(\mathcal{H}) \mid v = (i\lambda, z_2, \ldots)\} \cap \{A \in \mathcal{L}^a(\mathcal{H}) \mid (Ae_1, e_1) = 0\}$$
$$= \{T_v \in \mathcal{L}^a(\mathcal{H}) \mid v = (0, z_2, z_3, \ldots)\}$$

which is nothing but the *canonical decomposition* at the identity of $\mathcal{U}(\mathcal{H})$. Now $H' \cap K'$ is closed since $H' \cap K' = \ker(P_{H \oplus K})$ (see Appendix). By translations, $i \cdot \iota$ is an injective immersion. Finally, let us prove that $i \cdot \iota$ is closed: if C_1 is closed in $\mathcal{U}(\mathcal{H})_{e_1}$ and C_2 is closed (and then compact) in S^1, then C_1 is closed in $\mathcal{U}(\mathcal{H})$ since $\mathcal{U}(\mathcal{H})_{e_1}$ is closed in $\mathcal{U}(\mathcal{H})$ and $\iota(C_2)$ is closed and compact in $\mathcal{U}(\mathcal{H})$. Then using the result that if A and B are closed subsets of a topological group G and B is compact, then AB is closed, we have that $i \cdot \iota = \mu \circ (i \times \iota)$ is closed. One has the homeomorphism

$$i \tilde{\cdot} \iota: \quad \mathcal{U}(\mathcal{H})_{e_1} \times S^1 \to \mathcal{U}(\mathcal{H})_{[e_1]}, \qquad i \tilde{\cdot} \iota(T, w) = i \cdot \iota(T, w) \quad \text{QED}$$

Corollary. By Theorem A (see Appendix), $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$ has a unique differentiable structure such that π: $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$ is a submersion. Moreover, the restriction $D\pi(Id)|_{H' \cap K'}$: $H' \cap K' \to T_{[Id]}\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$ is an isomorphism of topological vector spaces.

Remark. Through $\bar{\beta}_{[e_1]}$ the differentiable structure on $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$ passes to $\mathcal{P}(\mathcal{H})$; in particular,

$$D\beta_{[e_1]}(Id)|_{H' \cap K'} = D\bar{\beta}_{[e_1]}([Id]) \circ D\pi(Id)|_{H' \cap K'}: \quad H' \cap K' \to T_{[e_1]}\mathcal{P}(\mathcal{H})$$

is a topological vector space isomorphism. From $\beta_{[e_1]} = q \circ \alpha_{e_1}$ we have

$$D\beta_{[e_1]}(Id) = Dq(e_1) \circ D\alpha_{e_1}(Id): \quad \mathcal{L}^a(\mathcal{H}) \to T_{[e_1]}\mathcal{P}(\mathcal{H}), \quad T \mapsto Dq(e_1)(Te_1)$$

and therefore $D\beta_{[e_1]}(Id)(T_v) = Dq(e_1)(v)$.

4.6. ξ_C as a Smooth Principal Bundle

We shall prove that the projective Hilbert bundle ξ_C: $S^1 \to S(\mathcal{H}) \xrightarrow{q} \mathcal{P}(\mathcal{H})$ is a smooth principal S^1-bundle. We shall use the following facts: $S(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ are homogeneous spaces of the Banach Lie group $\mathcal{U}(\mathcal{H})$ and therefore ξ_C is diffeomorphic to the bundle

$$S^1 \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} \xrightarrow{\bar{q}} \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$$

with $\bar{q}$ given by the composition

$$\begin{array}{ccc} S(\mathcal{H}) & \xrightarrow{q} & \mathcal{P}(\mathcal{H}) \\ {\scriptstyle \bar{\alpha}_{e_1}} \big\uparrow & & \big\uparrow {\scriptstyle \bar{\beta}_{[e_1]}} \\ \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{e_1} & \xrightarrow[\bar{q}]{} & \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]} \end{array}$$

i.e., $\bar{q} = \bar{\beta}_{[e_1]}^{-1} \circ q \circ \bar{\alpha}_{e_1}$ (see Sections 4.3 and 4.5); a surjective submersion has local sections and its consequence (see Section 4.3); the smooth functions p and π defined, respectively, in Sections 4.3 and 4.5 and satisfying $\bar{\alpha}_{e_1} \circ p = \alpha_{e_1}$ and $\bar{\beta}_{[e_1]} \circ \pi = \beta_{[e_1]}$ are surjective submersions. For simplicity of notation we shall write $\mathcal{U}(\mathcal{H}) \equiv G$, $\mathcal{U}(\mathcal{H})_{[e_1]} \equiv H$, and $\mathcal{U}(\mathcal{H})_{e_1} \equiv K$; if $T \in \mathcal{U}(\mathcal{H})$, then $TK \equiv [T]_K$ and $TH \equiv [T]_H$. Then $\bar{q}([T]_K) = [T]_H$ and for any $[T]_H$ there is a diffeomorphism $\bar{q}^{-1}(\{[T]_H\}) \to H/K$. We have:

(i) The commutative diagram

$$\begin{array}{ccc} & G & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle \pi} \\ G/K & \xrightarrow[\bar{q}]{} & G/H \end{array}$$

implies that $\bar{q}$ *is smooth.*

(ii) $G \xrightarrow{\pi} G/H$ has smooth local sections; then for any $[T]_H \in G/H$ there is an open neighborhood $U \subset G/H$ and a *smooth function* λ: $U \to G$ satisfying $\pi \circ \lambda = id_U$.

(iii) The inclusion ι: $H \to G$ induces the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\iota} & G \\ {\scriptstyle pr} \big\downarrow & & \big\downarrow {\scriptstyle p} \\ H/K & \xrightarrow[\bar{\iota}]{} & G/K \end{array}$$

$p \circ \iota$ is smooth and the projection pr is a surjective submersion; therefore $\bar{\iota}$ *is smooth*, where $\bar{\iota}(TK) = [T]_K$.

(iv) The product in G induces the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle id_G \times p} \big\downarrow & & \big\downarrow {\scriptstyle p} \\ G \times G/K & \xrightarrow[\bar{\mu}]{} & G/K \end{array}$$

where $\overline{\mu}(T', [T]_K) = [T'T]_K$ is the induced action. Since $p \circ \mu$ is smooth and $id_G \times p$ is a surjective submersion, then $\overline{\mu}$ *is smooth.*

(v) From (ii)–(iv) the composition $\tilde{\varphi} := \overline{\mu} \circ (\lambda \times \bar{\iota})$ given by the diagram

$$\begin{array}{ccc} & G/K & \\ {}^{\tilde{\varphi}}\nearrow & & \nwarrow^{\overline{\mu}} \\ U \times H/K & \xrightarrow[\lambda \times \bar{\iota}]{} & G \times G/K \end{array}$$

is *smooth.* One has

$$\begin{aligned} \tilde{\varphi}([T]_H, T'K) &= \overline{\mu} \circ (\lambda \times \bar{\iota})([T]_H, T'K) \\ &= \overline{\mu}(\lambda([T]_H), [T]_K) = [\lambda([T]_H)T']_K \end{aligned}$$

(vi) Defining

$$\varphi: U \times H/K \to \overline{q}^{-1}(U), \qquad \varphi([T]_H, T'K) := \tilde{\varphi}([T]_H, T'K)$$

φ is smooth, the diagram

$$\begin{array}{ccc} U \times H/K & \xrightarrow{\varphi} & \overline{q}^{-1}(U) \\ & {}_{pr_1}\searrow \quad \swarrow_{\overline{q}|} & \\ & U & \end{array}$$

commutes, and

$$\psi: \overline{q}^{-1}(U) \to U \times H/K, \qquad \psi([T]_K) := ([T]_H, \lambda([T]_H)^{-1}[T]_K)$$

is a smooth inverse of φ. Now, ψ is smooth since its first coordinate is $\overline{q}$ and the second coordinate ψ_2 is given by the composition $\overline{\mu} \circ (inv \circ \lambda \circ \overline{q}, \nu)$, where inv is the inverse in G and ν: $\overline{q}^{-1}(U) \to G/K$ is the inclusion, i.e., one has the commutative diagram

$$\begin{array}{ccc} & G/K & \\ {}^{\psi_2}\nearrow & & \nwarrow^{\overline{\mu}} \\ \overline{q}^{-1}(U) & \xrightarrow[(inv \circ \lambda \circ \overline{q}, \nu)]{} & G \times G/K \end{array}$$

ψ is a right inverse of φ since

$$\varphi \circ \psi([T]_K) = \varphi([T]_H, \lambda([T]_H)^{-1}[T]_K)$$
$$= \tilde{\varphi}([T]_H, [\lambda([T]_H)^{-1}T]_K) = [\lambda([T]_H)\lambda([T]_H)^{-1}T]_K = [T]_K$$

similarly one proves that ψ is a left inverse of φ.

(vii) From Section 4.5, H/K is diffeomorphic to S^1. Then the diffeomorphism $\{\varphi: U \times S^1 \to \overline{q}^{-1}(U)\}$ corresponding to an open cover $\{U\}$ of G/H is a set of local trivializations of ξ_{C}. QED

4.7. The Universal Connection and Its Complex Structure

In Section 4.4 we obtained the canonical decomposition of the tangent spaces to the unitary group of the Hilbert space $\mathcal{H} = l^2(\mathrm{N})$, $T_A\mathcal{U}(\mathcal{H}) = H_A \oplus_{\mathbb{R}} V_A$, with horizontal and vertical subspaces, respectively, given by $H_A = \{A \circ T_v \mid v = (0, z_2, \ldots) \in \mathcal{H}\}$ and $V_A = \{A \circ T_v \mid v = (i\lambda, 0, \ldots) \in \mathcal{H}\}$, where $\mathcal{L}^a(\mathcal{H}) \ni T_v: \mathcal{H} \to \mathcal{H}$ is given by $T_v(h) = h_1 v + (h_1\bar{z}_1 - (h, v))e_1$ (see Section 4.3); notice that $\lambda T_v = T_{\lambda v}$ for $\lambda \in \mathbb{R}$. (In particular, at the identity, $T_{Id}\mathcal{U}(\mathcal{H}) = H_{Id} \oplus_{\mathbb{R}} V_{Id}$ with $H_{Id} = \{T_v \mid v = (0, z_2, \ldots) \in \mathcal{H}\}$ and $V_{Id} = \{T_v \mid v = (i\lambda, 0, \ldots) \in \mathcal{H}\}$.) Since ξ_{C} is a smooth principal S^1-bundle with horizontal spaces given by the inner product in $\mathcal{H}$ and the subspaces H_A are obtained from them by $D\overline{\alpha}_{e_1}$, then the family $\{H_A\}$ is smooth. It can be easily shown that if one defines the left action $L: S^1 \times \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ by $L(\omega, B) := L_{\iota(\omega)}(B)$, then $H_{L_{\iota(\omega)}(A)} = DL_{\iota(\omega)}(A)(H_A)$, i.e., the family $\{H_A\}$ is left S^1-equivariant. (To have right S^1-equivariance—according to Section 2.2—one should take the elements of the homogeneous spaces G/H as $[g] = Hg$ instead of $[g] = gH$, throughout this paper.) Therefore the subspaces $\{H_A\}$ define a connection. The connection is universal since its restriction to each odd-dimensional sphere coincides with the Narasimhan–Ramanan connection, and it is responsible for the geometric phase in the infinite-dimensional case.

Finally, we prove that at each $A \in \mathcal{U}(\mathcal{H})$, H_A has a *complex structure*: define

$$J_A: \quad H_A \to H_A, \qquad A \circ T_v \mapsto J_A(A \circ T_v) := A \circ T_{iv}$$

Then

$$J_A(A \circ T_{v_1} + A \circ T_{v_2}) = J_A(A \circ (T_{v_1} + T_{v_2})) = J_A(A \circ T_{v_1+v_2})$$
$$= A \circ T_{i(v_1+v_2)} = A \circ T_{iv_1+iv_2} = A \circ (T_{iv_1} + T_{iv_2})$$
$$= A \circ T_{iv_1} + A \circ T_{iv_2}$$
$$= J(A \circ T_{v_1}) + J(A \circ T_{v_2}),\ J_A(\lambda A \circ T_v)$$
$$= J_A(A \circ \lambda T_v) = J_A(A \circ T_{\lambda v}) = A \circ T_{i\lambda v} = A \circ T_{\lambda iv}$$
$$= A \circ \lambda T_{iv} = \lambda A \circ T_{iv} = \lambda J_A(A \circ T_v)$$

i.e., J_A is R-linear and

$$J_A^2(A \circ T_v) = J_A(A \circ T_{iv}) = A \circ T_{i(iv)} = A \circ T_{-v} = -A \circ T_v$$

i.e., $J_A^2 = -id_{H_A}$. With the complex structure, H_A becomes a complex vector space with product by complex numbers given by $z(A \circ T_v) = A \circ T_{zv}$, i.e., $i(A \circ T_v) = J_A(A \circ T_v)$. It is clear that the complex structure carried by the universal connection is *canonical*. This reinforces the statement made at the beginning of this work, that quantum mechanics makes use of complex numbers in an essential way, since it is precisely the universal connection that drives the time evolution of the geometrical part of the wave function.

The isomorphism of topological vector spaces given by the restriction to H_A of the differential of π: $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}$ at A, i.e., $D\pi(A)|_{H_A}$: $H_A \to T_{[A]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]})$, allows us to translate the complex structure from the tangent spaces to $\mathcal{U}(\mathcal{H})$ to the tangent spaces to $\mathcal{P}(\mathcal{H})$, namely

$$J_{[A]}: \quad T_{[A]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]}) \to T_{[A]}(\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{[e_1]})$$

$$J_{[A]}(t_{[A]}) := D\pi(A)|_{H_A}(J_A(D\pi(A)|_{H_A}^{-1}(t_{[A]})))$$

APPENDIX

Definition. An equivalence relation $\sim$ on a smooth manifold M is called *regular* if the quotient space $M/\sim$ carries a differentiable structure such that the projection p: $M \to M/\sim$ is a submersion. If $\sim$ is a regular equivalence relation, then $M/\sim$ is called the quotient manifold.

Proposition. Let $\sim$ be a regular equivalence relation on M. Then:

(i) Any smooth map g: $M \to N$ compatible with $\sim$, i.e., $x_1 \sim x_2$ implies $g(x_1) = g(x_2)$, defines a unique smooth map $\tilde{g}$: $M/\sim \to N$ such that $\tilde{g} \circ p = g$.

(ii) The manifold structure on $M/\sim$ is unique (up to diffeomorphism).

Definition. Let $\sim$ be an equivalence relation on M. The set $\Gamma(\sim) = \{(x_1, x_2) \in M \times M \mid x_1 \sim x_2\}$ is called the *graph* of $\sim$.

Theorem (Godement) (Abraham *et al.*, 1988). An equivalence relation $\sim$ on a smooth manifold M is regular if and only if:

(i) $\Gamma(\sim)$ is a regular submanifold of $M \times M$.

(ii) p_1: $\Gamma(\sim) \to M$, where $p_1(x_1, x_2) = x_1$ is a submersion.

Theorem A. Let G be a Banach Lie group, and $H \subset G$ a subgroup which is a regular submanifold of G. Then the action $G \times H \to G$ defines a regular equivalence relation on G. Hence G/H has a unique smooth manifold structure such that p: $G \to G/H$ is a submersion. Moreover, if $L \subset T_gG$ is a closed complement of ker $Dp(g)$, then $Dp(g)|_L$: $L \to T_{[g]}G/H$ is an isomorphism.

Proof. In this case the equivalence relation on G is given by $g_1 \sim g_2$ if and only if there exists $h \in H$ such that $g_1 h = g_2$. Define γ: $G \times H \to G \times G$ by $\gamma(g, h) = (g, gh)$. Clearly Image $\gamma = \Gamma(\sim)$. Now, γ is smooth since it is a projection on the first coordinate and the composition of an inclusion and the product in G: $G \times G \to G \times G \to G$ on the second. Assume $\gamma(g_1, h_1) = \gamma(g_2, h_2)$; then $(g_1, g_1 h_1) = (g_2, g_2 h_2)$, so that $g_1 = g_2$ and $h_1 = h_2$. Therefore γ is injective.

One can easily show that the differential of γ at $(e, e) \in G \times H$, $D\gamma(e, e)$: $T_eG \times T_eH \to T_eG \times T_eG$ is given by $D\gamma(e, e)(v, w) = (v, v + w)$, where $T_eH \subset T_eG$, which is clearly a monomorphism. Now we will show that the image of $D\gamma(e, e)$ is a split subspace of $T_eG \times T_eG$. Let f: $T_eG \times T_eG \to T_eG$ be given by $f(v_1, v_2) = v_2 - v_1$, which is continuous since T_eG is a topological vector space. Since H is a regular submanifold of G, T_eH is a closed split subspace of T_eG; therefore $f^{-1}(T_eG - T_eH)$ is open, but $f^{-1}(T_eG - T_eH) = (T_eG \times T_eG) -$ Image $D\gamma(e, e)$, hence Image $D\gamma(e, e)$ is closed. Let $P' \in L(T_eG)$ be the projection operator for the split subspace T_eH. Define P: $T_eG \times T_eG \to T_eG \times T_eG$ by $P(v_1, v_2) = (v_1, v_1 + P'(v_2 - v_1))$. Since P' is continuous and linear and $P' \circ P' = P'$, one can show that P is continuous and linear and $P \circ P = P$. We also have that Image $D\gamma(e, e) = \{(v_1, v_2) | P(v_1, v_2) = (v_1, v_2)\}$. Therefore $T_eG \times T_eG =$ Image $D\gamma(e, e) \oplus \ker P$, hence γ is an immersion at (e, e). To show that γ is an immersion at any point $(g, h) \in G \times H$, consider the following commutative diagram:

$$\begin{array}{ccc} G \times H & \xrightarrow{\gamma} & G \times G \\ {\scriptstyle L_g \times R_h}\downarrow & & \downarrow{\scriptstyle L_g \times (R_h \circ L_g)} \\ G \times H & \xrightarrow[\gamma]{} & G \times G \end{array}$$

Since L_g and R_h are diffeomorphisms, taking differentials, we can show that $D\gamma(g, h)$ is a monomorphism whose image is a split subspace.

Let ψ: $G \times G \to G \times G$ be given by $\psi(g_1, g_2) = (g_1, g_1^{-1} g_2)$. This map is clearly smooth, and in particular its restriction to $\gamma(G \times H) \subset G \times G$ is continuous. Since $H \subset G$ has the subspace topology, $\psi|_{\gamma(G \times H)}$: $\gamma(G \times H) \to G \times H$ is continuous. Since this map is the inverse of γ, then γ is a topological embedding, i.e., γ is a homeomorphism onto its image. Therefore γ is a regular embedding and then $\Gamma(\sim)$ is a regular submanifold of $G \times G$.

Now we shall show that p_1: $G \times G \to G$ restricted to $\gamma(G \times H)$ is a submersion. Consider

$$\begin{array}{ccc} T_eG \times T_eH & \xrightarrow{D\gamma(e, e)} & T_eG \times T_eG \\ & & \downarrow{\scriptstyle Dp_1(e, e)} \\ & & T_eG \end{array}$$

Let $v \in T_eG$; then $(v, v) = D\gamma(e, e)(v, 0)$ and $Dp_1(e, e)(v, v) = v$, so $Dp_1(e, e)|_{\text{Image } D\gamma(e,e)}$ is surjective. Its kernel is the subspace $\{0\} \times T_eH$. Since T_eG is Hausdorff, the diagonal subspace $\Delta = \{(v, v)|v \in T_eG\}$ is closed and we clearly have that $\Delta \cap \{0\} \times T_eH = \{0\}$ and $\Delta \oplus \{0\} \times T_eH =$ Image $D\gamma(e, e)$. To see that $p_1|_{\text{Image } \gamma}$ is a submersion at any other point $(g, gh) \in$ Image γ, we consider the following commutative diagram and take the differential of each map:

$$\begin{array}{ccccc}
G \times H & \xrightarrow{\gamma} & G \times G & \xrightarrow{p_1} & G \\
\Big\downarrow{\scriptstyle L_g \times R_h} & & \Big\downarrow{\scriptstyle L_g \times (R_h \circ L_g)} & & \Big\downarrow{\scriptstyle L_g} \\
G \times H & \xrightarrow[\gamma]{} & G \times G & \xrightarrow[p_1]{} & G
\end{array}$$

Therefore by Godement's theorem G/H has a unique smooth structure such that p: $G \to G/H$ is a submersion. To prove the second statement of the theorem, let φ: $V \to W$ be a surjective, continuous, linear map between Banach spaces. If L is a closed subspace of V such that $L \oplus \ker \varphi = V$, then $\varphi|_L$: $L \to W$ is an isomorphism. To verify this, take $l \in L$ such that $\varphi(l) = 0$; then $l \in L \cap \ker \varphi = \{0\}$, so $l = 0$; now, since φ is surjective, given $w \in W$, there exists $v \in V$ such that $\varphi(v) = w$, but $v = l + a$, where $a \in \ker \varphi$; therefore $\varphi(v) = \varphi(l) = w$, so $\varphi|_L$ is surjective. Since $\varphi|_L$ is a continuous linear isomorphism, by Banach's isomorphism theorem, $\varphi|_L$ is a homeomorphism.

The result now follows from the fact that for each $g \in G$, $Dp(g)$: $T_gG \to T_{[g]}G/H$ is surjective and its kernel splits. QED

Proposition B. Let V be a Banach space and H and K closed split subspaces with $\dim K < \infty$ and such that $H \cap K = \{0\}$. If $H \oplus H' = V$, $K \oplus K' = V$, $H \subset K'$, and $K \subset H'$, then $H \oplus K$ is a closed split subspace of V and $V = (H \oplus K) \oplus (H' \cap K')$.

Proof. (i) From Abraham *et al.* (1988) if H and K are closed and *dim* $K < \infty$, then $H \oplus K$ is closed.

(ii) There exist linear continuous operators P_H, P_K: $V \to V$ satisfying

$$P_H^2 = P_H$$

$$P_K^2 = P_K$$

$$V = \{\text{fixed vectors of } P_H\} \oplus \ker P_H = \{\text{fixed vectors of } P_K\} \oplus \ker P_K$$

Then if $v \in V$, $v = h \oplus h'$ with $P_H(v) = h$ and $v = k \oplus k'$ with $P_K(v) = k$. Defining $P_{H\oplus K}$: $V \to V$, $P_{H\oplus K} := P_H \oplus P_K$, we have:

(a) $P_{H\oplus K} \circ P_{H\oplus K}(v) = P_{H\oplus K}(P_H(v) \oplus P_K(v)) = P_{H\oplus K}(P_H(v)) \oplus P_{H\oplus K}(P_K(v)) = P_H^2(v) \oplus P_K(P_H(v)) \oplus P_H(P_K(v)) \oplus P_K^2(v) = P_H(v) \oplus P_K(v) = (P_H \oplus P_K)(v)$ since $P_K(h) = P_H(k) = 0$ because $h \in H \subset K'$ and $k \in K \subset H'$.

(b) Let $v \in H \oplus K$; then $P_{H\oplus K}(v) = P_{H\oplus K}(h \oplus k) = P_H(h) \oplus P_K(h) = h \oplus k$, i.e., $H \oplus K \subset \{\text{fixed vectors of } P_{H\oplus K}\}$; let $v \in V$ and $P_{H\oplus K}(v) = P_H(v) \oplus P_K(v) = v$, since $P_H(v) = h$ and $P_K(v) = k$; then $v = h \oplus k$, i.e., $\{\text{fixed vectors of } P_{H\oplus K}\} \subset H \oplus K$, and therefore $\{\text{fixed vectors of } P_{H\oplus K}\} = H \oplus K$. Then $V = (H \oplus K) \oplus \ker(P_{H\oplus K})$ with $\ker(P_{H\oplus K}) = \{v \in V | h \oplus k = 0\}$.

(iii) Let $v \in H' \cap K'$, i.e., $v = 0 \oplus h' = 0 \oplus k'$; then $h \oplus k = 0$ since $h = k = 0$, i.e., $H' \cap K' \subset \ker(P_{H\oplus K})$; let $h = -k$; then $h = k = 0$ since $H \cap K = \{0\}$ and so $v = h' = k'$, i.e., $\ker(P_{H\oplus K}) \subset H' \cap K'$. Then $\ker(P_{H\oplus K}) = H' \cap K'$. QED

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